SOME LIMIT THEOREMS FOR WEIGHTED SUMS OF RANDOM VARIABLES

BY

ANDRÉ BRUCE ADLER

A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN
PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1987

ACKNOWLEDGMENTS

I would like to thank Dr. Andrew Rosalsky, my dissertation advisor, not only for his overwhelming assistance, but also for his friendship. Also, I would like to thank Dr. Malay Ghosh, Dr. Ronald Randles, and Dr. Murali Rao for serving on my committee. For serving on my qualifying examination and oral defense committees, I would like to thank Dr. Rocco Ballerini. In addition, I would like to thank Ms. Cindy Zimmerman for her expert typing of this dissertation.

I would also like to thank my parents for their support which helped me in reaching my goals.

Finally, I would like to thank Dawn Peters who was always there when I needed her.

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Abstract of Dissertation Presented to the Graduate School
of the University of Florida in Partial Fulfillment of the
Requirements for the Degree of Doctor of Philosophy

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BY

ANDRÉ BRUCE ADLER

May, 1987

Chairman: Andrew Rosalsky
Major Department: Statistics

Asymptotic behavior of normed weighted sums of the form $W_n = \sum_{k=1}^{n} a_k(X_k - \gamma_k)/b_n$ is studied. Central limit theorems as well as strong and weak laws of large numbers are obtained.

Firstly, we establish a generalized central limit theorem $W_n \xrightarrow{d} N(0,1)$ assuming the $\{X, X_n, n \geq 1\}$ are independent, identically distributed with $EX = 0$, $EX^2 = \infty$. The truncated second moment is assumed to be slowly varying at infinity. Then, via a transformation, we obtain a similar result where the condition $EX = 0$ is removed. The norming sequence $\{b_n, n \geq 1\}$ is defined in terms of the common distribution, and conditions which elicit an explicit asymptotic representation for $\{b_n, b \geq 1\}$ are found.

Several generalized strong laws of large numbers ($W_n \to 0$ almost surely) are proved. In some of them, the random variables

$\{X_n, n \geq 1\}$ are stochastically dominated by a random variable X, while in others the random variables $\{X_n, n \geq 1\}$ are independent, identically distributed. Two theorems are proved showing that the assumption of independence is, in general, not always needed to obtain a strong law. More specifically, their hypotheses involve both the behavior of the tail of the (marginal) distributions of the $\{X_n, n \geq 1\}$ and the growth behavior of the constants b_n/a_n. As special cases, both old and new results are obtained. Moreover, for independent, identically distributed $\{X, X_n, n \geq 1\}$, a strong law is established under the assumption of regular variation of the tail $P\{|X| > x\}$ of common distribution function.

The famous Petersburg paradox is also examined in more general terms. It is shown that $P\{\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} a_k X_k/b_n = c\} = 0$ for any sequence $\{b_n, n \geq 1\}$ and finite nonzero constant c whenever the random variables $\{X, X_n, n \geq 1\}$ are independent, identically distributed with $E|X| = \infty$ and $|a_n| \uparrow$.

Finally, a generalized weak law of large numbers ($W_n \xrightarrow{P} 0$) is obtained. Using this theorem we are able to show that the modified Petersburg game does indeed have a solution in the "classical" or "weak" sense, i.e., there exists a sequence $\{b_n, n \geq 1\}$ such that $\sum\limits_{k=1}^{n} a_k X_k/b_n \xrightarrow{P} c$, where the requirements placed on $\{X, X_n, n \geq 1\}$ and c are as before.

Throughout, we present examples that illustrate our results and compare them with others from here and elsewhere.

CHAPTER ONE
INTRODUCTION

This dissertation will explore three major modes of convergence studied in probability theory. We will investigate both the weak and strong limiting behavior of a normed sum of weighted random variables. Such sequences of normed sums are expressed in the form

$$\frac{\sum_{k=1}^{n} a_k(X_k-\gamma_k)}{b_n}, \quad n \geq 1. \tag{1}$$

While the sequences $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ will always be numerical, the sequence $\{X_n, n \geq 1\}$ will consist of random variables which may or may not be independent or even identically distributed. The sequence $\{\gamma_n, n \geq 1\}$ will take on many different forms, ranging from conditional expectations to all zeros.

Chapter Two will examine when (1) converges in distribution to a standard normal random variable. This is commonly referred to as a central limit theorem (CLT). A sequence of random variables, say Z_n, converges in distribution to a standard normal random variable if

$$P\{Z_n \leq z\} \to \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}}\exp\{-\tfrac{1}{2}t^2\}dt, \quad -\infty < z < \infty.$$

The case when the random variables $\{X_n, n \geq 1\}$ are independent with finite second moment has been thoroughly investigated (see Theorem 2.1). The topic of interest in Chapter Two is whether this limiting behavior still prevails when the second moment is infinite. We will only consider the situation in which the random variables are independent, identically distributed (i.i.d.). The first major result will establish a CLT for mean zero random variables. In contrast to the case when the second moment is finite where the norming constants are universal (up to a multiplicative constant), the norming constants $\{b_n, n \geq 1\}$ when the second moment is infinite are not universal but, rather, depend on the common distribution of the $\{X_n, n \geq 1\}$. Then by studying the shifted sequence of random variables, $Y_n = X_n - EX_n$, $n \geq 1$, we will obtain a CLT for random variables with arbitrary first moment and infinite second moment.

Strong laws of large numbers (SLLN) are investigated in Chapter Three. The sequence in (1) is said to obey the strong law of large numbers if $\sum_{k=1}^{n} a_k(X_k - \gamma_k)/b_n \to 0$ almost surely (a.s.), that is,

$$P\{\lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_k(X_k-\gamma_k)}{b_n} = 0\} = 1.$$

This chapter contains many generalizations of the classical Kolmogorov SLLN ($\sum_{k=1}^{n} X_k/n \to 0$ a.s. whenever $\{X_n, n \geq 1\}$ are i.i.d. with mean 0). Throughout, we will extend and generalize previous results thereby obtaining them as special cases or corollaries of our work. See Stout (1974, Chapter 4) for an excellent survey of known results on the SLLN problem for weighted sums of random variables.

Mathematicians have, over the centuries, tried to understand the Petersburg paradox. Given a game with X_k winnings at the k^{th} stage, it was asked if the game could be made "fair" in the sense that there is some entrance fee, $b_k - b_{k-1}$, $k \geq 2$, such that $\sum_{k=1}^{n} X_k/b_n \to 1$ a.s. This is relatively easy when $\{X, X_n, n \geq 1\}$ are i.i.d. with $E|X| < \infty$, but is a source of confusion when $E|X| = \infty$. This problem, when the random variables are not integrable, is called the Petersburg paradox.

Placing mild restrictions on the sequence $\{a_n, n \geq 1\}$, we will prove that

$$P\{\lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_k X_k}{b_n} = c\} = 0,$$

for every sequence $\{b_n, n \geq 1\}$ and every finite nonzero constant c provided that the random variables $\{X, X_n, n \geq 1\}$ are i.i.d. with $E|X| = \infty$. This result generalizes one of Chow and Robbins (1961) wherein $a_k \equiv 1$. For a detailed discussion of this paradox, see Feller (1968, p. 251-253). Feller, however, does produce a sequence $\{b_n, n \geq 1\}$ such that $\sum_{k=1}^{n} X_k/b_n \xrightarrow{P} 1$ for a specific sequence of i.i.d. random variables with $E|X| = \infty$.

There is also the question as to whether a generalized law of the iterated logarithm can hold when $EX^2 = \infty$, i.e.,

$$P\left\{\limsup_{n\to\infty} \frac{\sum_{k=1}^{n} a_k X_k}{b_n} = 1\right\} = 1.$$

See Rosalsky (1981) for some results of this type.

In the final chapter, we consider the weak law of large numbers (WLLN). The normed sum (1) is said to obey the weak law of large numbers if $\sum_{k=1}^{n} a_k(X_k-\gamma_k)/b_n \xrightarrow{P} 0$, that is,

$$P\left\{\left|\frac{\sum_{k=1}^{n} a_k(X_k-\gamma_k)}{b_n}\right| > \varepsilon\right\} = o(1) \text{ for all } \varepsilon > 0.$$

Clearly, if a SLLN holds, then a WLLN also holds for the same sequences. Hence Chapter Three also establishes many weak laws. In view of a very general result known as the Degenerate Convergence Criterion (see, e.g., Chow and Teicher, 1978, p. 329), we acknowledge the fact that most work on the WLLN for independent $\{X_n, n \geq 1\}$ has already been done.

Research in the central limit theorem and laws of large numbers date back to the 18th century. The very first SLLN was proved in 1909 by Émile Borel, while the WLLN dates back much earlier. A WLLN for i.i.d. random variables with a finite second moment was established by Jacob Bernoulli and published posthumously by his nephew Nicholas in 1713. By a simple application of Chebyshev's inequality it was shown that $\sum_{k=1}^{n} X_k/n \xrightarrow{P} 0$ when $\{X, X_n, n \geq 1\}$ are i.i.d. mean zero random variable with $EX^2 < \infty$. Many famous mathematicians have played a prominent role in the evolution of the

CLT. For a detailed history of the development of these and related concepts, see Feller (1945) and Le Cam (1986). Also, for an excellent survey of known results, see Petrov (1975).

Some remarks about notation are in order. Throughout, the symbol C will denote a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance. Also, in order to avoid minor complications, it proves convenient to define $\log x = \log_e \max\{e,x\}$, $x \geq 0$, where $\log_e t$ denotes the natural logarithm. Finally, for $x \geq 0$, $\log_2 x$ will be used to denote $\log \log x$.

CHAPTER TWO
GENERALIZED CENTRAL LIMIT THEOREMS

2.1 Introduction

The question as to whether a sequence of random variables obeys a CLT has a long and rich history. Firstly, in the 1700s, there was DeMoivre and Laplace discovering a primitive version of the CLT for a sequence of Bernoulli random variables. Two hundred years later, in the 1930s, Lindeberg, Lévy, and Feller established what we now generally refer to as the CLT. This famous result, which we will now state as Theorem 2.1, is known as the Lindeberg-Feller theorem.

Theorem 2.1 (Chow and Teicher, 1978, p. 291). If $\{X_n, n \geq 1\}$ are independent random variables with $EX_n = 0$, $Var(X_n) = \sigma_n^2$, $n \geq 1$, satisfying

$$\sum_{k=1}^{n} EX_k^2 I(|X_k| > \varepsilon s_k) = o(s_n^2) \tag{1}$$

for all $\varepsilon > 0$, where $s_n^2 = \sum_{k=1}^{n} \sigma_k^2$, $n \geq 1$, then

$$\frac{\sum_{k=1}^{n} X_k}{s_n} \xrightarrow{d} N(0,1). \tag{2}$$

Conversely, if (2) holds and $\sigma_n = o(s_n)$, $s_n \to \infty$ then (1) obtains.

Proof. See Chow and Teicher (1978, p. 290-293).□

Clearly, this theorem is not applicable when $\sigma_n^2 = \infty$ for some $n \geq 1$. However, Lévy (1935), Khintchine (1935), and Feller (1935) have studied the CLT problem when the random variables $\{X_n, n \geq 1\}$ are i.i.d. with $EX_1^2 = \infty$. Their version of the CLT may be stated as follows.

Theorem 2.2 (Chow and Teicher, 1978, p. 300). If $\{X, X_n, n \geq 1\}$ are i.i.d. random variables with $EX^2 = \infty$, then

$$\frac{\sum_{k=1}^{n} X_k - A_n}{B_n} \xrightarrow{d} N(0,1)$$

for some $B_n > 0$ and A_n iff

$$\lim_{c\to\infty} \frac{c^2 P\{|X| > c\}}{EX^2 I(|X| \leq c)} = 0.$$

Moreover, for all $n \geq 1$, B_n may be chosen as the supremum of all $c > 0$ so that $c^{-2}EX^2I(|X| < c) \geq \frac{1}{n}$, while A_n may be taken as

$$A_n = nEXI(|X| \leq B_n),\ n \geq 1.$$

Proof. See Chow and Teicher (1978, p. 300-302). □

Theorem 2.2 was first proved by using the function H in (4) while our extension will use the function G in (5) instead.

This chapter will look into the CLT problem when $\sigma_n^2 = \infty$, $n \geq 1$, for the weighted sum $\sum_{k=1}^{n} a_k X_k$. The CLT for $\{X_n, n \geq 1\}$ i.i.d. random variables, $EX_1 = 0$, $\mathrm{Var}(X_1) = 1$, $\{a_n, n \geq 1\}$ nonzero constants with $\sum_{k=1}^{n} a_k^2 \to \infty$ and $a_n^2 = o(\sum_{k=1}^{n} a_k^2)$ is an easy application of Theorem 2.1 (see, e.g., Chow and Teicher, 1978, p. 302).

2.2 Preliminary Lemmas

To study the CLT problem when $\sigma_n^2 = \infty$, one must examine a special class of functions. A function g is said to be slowly varying (at infinity) if for all $s > 0$, $g(sx) \sim g(x)$ as $x \to \infty$. Closely related to slowly varying functions are those that are regularly varying. A function h is said to be regularly varying with exponent ρ $(-\infty < \rho < \infty)$ if $h(x) = x^{\rho}g(x)$, for some slowly varying function g. A useful property of every positive slowly varying function is that (see Rosalsky, 1981)

$$\log g(x) = o(\log x) \text{ as } x \to \infty \tag{3}$$
$$\text{(and hence } g(x) = o(x^{\alpha}),\text{ all } \alpha > 0).$$

The question at hand is what condition should we place on the sequences of constants $\{a_n, B_n, e_n, n \geq 1\}$ and the i.i.d. random variables $\{X_n, n \geq 1\}$ to ensure that

$$\frac{\sum_{k=1}^{n} a_k(X_k - e_k)}{B_n} \xrightarrow{d} N(0,1)$$

when $\mathrm{Var}(X_1) = \infty$. As will become apparent, we need to know at which rate the variance of the variables approaches infinity. This is where the notion of slow variation will be applied. We define H, the truncated second moment of X, by

$$H(x) \equiv EX^2 I(|X| \leq x),\ x \geq 0. \tag{4}$$

Another function of interest is G defined by

$$G(x) \equiv \int_0^x 2tP\{|X| > t\}dt,\ x \geq 0. \tag{5}$$

The functions G and H were used by Rosalsky (1981) to prove a generalized law of the iterated logarithm for weighted sums with infinite variance.

The first lemma will be used to establish a relationship between H and G.

Lemma 2.1. For every random variable X and positive constant p,

$$p\int_0^s t^{p-1}P\{|X| > t\}dt = s^p P\{|X| > s\} + E|X|^p I(|X| \leq s)$$

for all $s \geq 0$.

Proof. Let $F_{|X|}$ denote the distribution function of the random variable $|X|$, that is, $F_{|X|}(t) = P\{|X| \leq t\}$. For $s \geq 0$, integration by parts yields

$$\int_0^s t^{p-1}P\{|X| > t\}dt = \int_0^s t^{p-1}(1-F_{|X|}(t))dt$$

$$= \frac{1}{p}[t^pP\{|X| > t\}\Big|_0^s + \int_0^s t^p dF_{|X|}(t)]$$

$$= \frac{1}{p}[s^pP\{|X| > s\} + E|X|^pI(|X| \leq s)]. \quad \square$$

Using this lemma, with p = 2, we obtain

$$G(x) = x^2P\{|X| > x\} + H(x), \ x \geq 0. \tag{6}$$

It should be noted that G is nondecreasing, continuous, and $G(x)/x^2 \downarrow 0$ as $x \uparrow \infty$ (see Rosalsky, 1981).

Other relationships between G and H which will be used throughout this chapter are given in the next lemma.

Lemma 2.2. (Rosalsky, 1981). The following are equivalent:

G is slowly varying,

H is slowly varying,

$G(x) \sim H(x)$,

$x^2P\{|X| > x\}/G(x) = o(1)$ as $x \to \infty$,

$x^2P\{|X| > x\}/H(x) = o(1)$ as $x \to \infty$.

At this point we note an interesting fact. If the random variable is not in $\mathcal{L}_2$, then by the following lemma it is in $\mathcal{L}_p$ for all $0 < p < 2$ provided H (or equivalently G) is slowly varying.

Lemma 2.3. If H is slowly varying, then $X \varepsilon \mathcal{L}_p$ for all $0 < p < 2$.

Proof. By Lemma 2.1 we obtain

$$E|X|^p = p\int_0^\infty t^{p-1}P\{|X| > t\}dt$$

whether or not $X \varepsilon \mathcal{L}_p$. This follows by considering the two cases; $E|X|^p < \infty$ or $E|X|^p = \infty$. Let $0 < \alpha < 2-p$. Then via (3), Lemma 2.2, and the hypothesis that H is slowly varying, there exists a number t_o such that if $t \geq t_o$, $H(t) \leq t^\alpha$ and $t^2P\{|X| > t\}/H(t) \leq 1$. Thus, $t^2P\{|X| > t\} \leq t^\alpha$ or $P\{|X| > t\} \leq t^{\alpha-2}$ whenever $t \geq t_o$. Hence

$$\begin{aligned} E|X|^p &= p\int_0^{t_o} t^{p-1}P\{|X| > t\}dt + p\int_{t_o}^\infty t^{p-1}P\{|X| > t\}dt \\ &\leq \int_0^{t_o} pt^{p-1}dt + p\int_{t_o}^\infty t^{p+\alpha-3}dt \\ &= t_o^p - \frac{p}{p+\alpha-2}t_o^{p+\alpha-2} < \infty, \end{aligned}$$

since $p+\alpha-2 < 0$. □

One thing to note is that slow variation of H ensures in particular that $E|X| < \infty$. This allows us to establish a lemma which takes advantage of Lemma 2.1.

Lemma 2.4. For any $\mathcal{L}_1$ random variable X and constant $c \geq 0$

$$E|X|I(|X| > c) = cP\{|X| > c\} + \int_c^\infty P\{|X| > t\}dt.$$

Proof. We apply Lemma 2.1 twice, first with $s = \infty$ and then $s = c$; in both cases we let $p = 1$. Therefore,

$$E|X|I(|X| > c) = E|X| - E|X|I(|X| \leq c)$$

$$= \int_0^\infty P\{|X| > t\}dt - [\int_0^c P\{|X| > t\}dt - cP\{|X| > c\}]$$

$$= cP\{|X| > c\} + \int_c^\infty P\{|X| > t\}dt. \quad \square$$

Before we proceed with our main result of this chapter, we conclude this section with one last lemma. Note that the hypothesis of Lemma 2.5 ensures, via (3), that $X \in \mathcal{L}_1$.

Lemma 2.5. If $P\{|X| > x\}$ is regularly varying with exponent $\rho < -1$, then

$$E|X|I(|X| > c) = (1+o(1))(\frac{\rho}{\rho+1})cP\{|X| > c\} \text{ as } c \to \infty.$$

Proof. We apply the following result in Feller (1971, p. 281): If $P\{|X| > x\}$ is regularly varying with exponent $\rho < -1$, then

$$\int_c^\infty P\{|X| > t\}dt = (1+o(1))(\frac{-c}{\rho+1})P\{|X| > c\} \text{ as } c \to \infty.$$

Using this and Lemma 2.4 we obtain

$$E|X|I(|X| > c) = cP\{|X| > c\} + \int_c^\infty P\{|X| > t\}dt$$

$$= cP\{|X| > c\} + (1+o(1))(\frac{-c}{\rho+1})P\{|X| > c\}$$

$$= (1+o(1))(\frac{\rho}{\rho+1})cP\{|X| > c\}. \quad \square$$

2.3 Mainstream

With these preliminaries accounted for, we are ready to state and prove our first major result, but first we need a version of Theorem 2.1 which establishes a CLT for triangular arrays.

Theorem 2.3. Suppose that for each n the random variables $X_{n1}, \ldots, X_{nr_n}$ are independent and satisfy $EX_{nk} = 0$, $\sigma_{nk}^2 = EX_{nk}^2$, $s_n^2 = \sum_{k=1}^{r_n} \sigma_{nk}^2$ for $1 \leq k \leq r_n$, $n \geq 1$. If

$$\lim_{n\to\infty} \sum_{k=1}^{r_n} \frac{1}{s_n^2} EX_{nk}^2 I(|X_{nk}| > \varepsilon s_n) = 0$$

holds for all $\varepsilon > 0$, then $\sum_{k=1}^{r_n} X_{nk}/s_n \xrightarrow{d} N(0,1)$.

Proof. See, e.g., Billingsley (1979, p. 310-312). $\square$

There is some notation that must be presented and will be used throughout the entire chapter. The functions G and H are defined as in (4) and (5), respectively. Let $Q(x) = x^2/G(x)$, $x > 0$, and since G is slowly varying, it is clear, via (3), that $Q(x) \to \infty$. It can also be shown that $Q(x) \uparrow \infty$ (see Rosalsky, 1981, Lemma 2). The sequence $\{q_n, n \geq 1\}$ is defined by $q_n = Q^{-1}(n)$, $n \geq 1$. Since $Q(x) \uparrow \infty$ we note that $q_n \uparrow \infty$. The next sequence of interest is $\{s_n^2, n \geq 1\}$ which is the sequence of partial sums of the squares of the weights, i.e., $s_n^2 = \sum_{k=1}^{n} a_k^2$, $n \geq 1$. Finally, we define the positive constants $\{B_n, n \geq 1\}$ via

$$B_n^2 = \frac{s_n^2}{n} q_n^2, \quad n \geq 1. \tag{7}$$

The first main result follows.

Theorem 2.4. Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables with $EX^2 = \infty$ and

$$\frac{x^2 P\{|X| > x\}}{H(x)} = o(1) \text{ as } x \to \infty. \tag{8}$$

Let $\{a_n, n \geq 1\}$ be constants such that

$$G\left(\frac{B_n}{\min\limits_{1 \leq k \leq n} |a_k|}\right) \sim G\left(\frac{B_n}{\max\limits_{1 \leq k \leq n} |a_k|}\right) \tag{9}$$

and

$$\sum_{k=1}^{n} P\{|X| > \varepsilon B_n / |a_k|\} = o(1) \text{ for all } \varepsilon > 0. \tag{10}$$

Then

$$\frac{\sum_{k=1}^{n} a_k X_k - A_n}{B_n} \xrightarrow{d} N(0,1) \tag{11}$$

where $\{B_n, n \geq 1\}$ is defined as in (7) and

$A_n = \sum_{k=1}^{n} a_k EXI(|X| \leq B_n/|a_k|), \; n \geq 1.$

Proof. Via Lemma 2.2, we see that (8) implies that G is slowly varying and that $G(x) \sim H(x) \to \infty$ (since $EX^2 = \infty$). Note that

$$G(q_n) = B_n^2/s_n^2 \tag{12}$$

since

$$n = Q(q_n) = q_n^2/G(q_n) \tag{13}$$

and hence $G(q_n) = \dfrac{q_n^2}{n} = \dfrac{B_n^2}{s_n^2}$ (by (7)).

Utilizing $G(x) \uparrow \infty$ and $q_n \to \infty$ we obtain via (12) that

$$s_n^2/B_n^2 = o(1). \tag{14}$$

Noting that

$$\frac{B_n}{\min\limits_{1 \leq k \leq n} |a_k|} = \frac{s_n q_n}{\sqrt{n} \min\limits_{1 \leq k \leq n} |a_k|} \geq q_n \geq \frac{s_n q_n}{\sqrt{n} \max\limits_{1 \leq k \leq n} |a_k|} = \frac{B_n}{\max\limits_{1 \leq k \leq n} |a_k|} \geq \frac{B_n}{s_n} \to \infty \tag{15}$$

and G is nondecreasing, it follows that

$$G\left(\frac{s_n q_n}{\sqrt{n}\min\limits_{1\le k\le n}|a_k|}\right) \ge G(q_n) \ge G\left(\frac{s_n q_n}{\sqrt{n}\max\limits_{1\le k\le n}|a_k|}\right),\ n \ge 1.$$

Using the fact that G is slowly varying together with (9) and (15) we obtain

$$G\left(\frac{s_n q_n}{\sqrt{n}\min\limits_{1\le k\le n}|a_k|}\right) \sim G(q_n) \sim G\left(\frac{s_n q_n}{\sqrt{n}\max\limits_{1\le k\le n}|a_k|}\right). \tag{16}$$

Next, we will show that

$$\frac{1}{B_n^2}\sum_{k=1}^{n} a_k^2 H(B_n/|a_k|) \to 1. \tag{17}$$

Since H is nondecreasing,

$$\frac{1}{B_n^2}\sum_{k=1}^{n} a_k^2 H(B_n/|a_k|) \ge \frac{1}{B_n^2}\sum_{k=1}^{n} a_k^2 H(B_n/\max_{1\le j\le n}|a_j|)$$

$$= \frac{s_n^2}{B_n^2} H\left(\frac{s_n q_n}{\sqrt{n}\max\limits_{1\le j\le n}|a_j|}\right) \qquad \text{(by (7))}$$

$$= (1+o(1))\frac{s_n^2}{B_n^2} G\left(\frac{s_n q_n}{\sqrt{n}\max\limits_{1\le j\le n}|a_j|}\right) \qquad \text{(by Lemma 2.2)}$$

$$= (1+o(1))\frac{s_n^2}{B_n^2}G(q_n) \qquad \text{(by (16))}$$

$$= 1+o(1) \qquad \text{(by (12)).}$$

Similarly,

$$\frac{1}{B_n^2}\sum_{k=1}^{n} a_k^2 H(B_n/|a_k|) \le \frac{1}{B_n^2}\sum_{k=1}^{n} a_k^2 H(B_n/\min_{1\le j\le n}|a_j|)$$

$$= \frac{s_n^2}{B_n^2}H\left(\frac{s_n q_n}{\sqrt{n}\min\limits_{1\le j\le n}|a_j|}\right) \qquad \text{(by (7))}$$

$$\le \frac{s_n^2}{B_n^2}G\left(\frac{s_n q_n}{\sqrt{n}\min\limits_{1\le j\le n}|a_j|}\right) \qquad \text{(by (6))}$$

$$= (1+o(1))\frac{s_n^2}{B_n^2}G(q_n) \qquad \text{(by (16))}$$

$$= 1+o(1) \qquad \text{(by (12)).}$$

Combining these two results yields (17).

Define $X_k^{(n)} = X_k I(|a_k X_k| \le B_n)$, $1 \le k \le n$, $n \ge 1$, and $S_n = \sum_{k=1}^{n} a_k X_k$, $S_n' = \sum_{k=1}^{n} a_k X_k^{(n)}$, $\gamma_n^2 = \mathrm{Var}(S_n')$, $n \ge 1$. Now via Lemmas 2.2 and 2.3 and (8) we note that $X \in \mathcal{L}_1$. Therefore,

$$\frac{1}{B_n^2}\sum_{k=1}^{n} a_k^2(EXI(|X| \le B_n/|a_k|))^2$$

$$\le \frac{1}{B_n^2}\sum_{k=1}^{n} a_k^2(E|X|)^2$$

$$= \frac{s_n^2}{B_n^2}(E|X|)^2 = o(1) \qquad \text{(by (14))}.$$

Using this result and (17) we conclude that $\gamma_n^2 \sim B_n^2$ as follows:

$$\gamma_n^2/B_n^2 = \frac{1}{B_n^2}\operatorname{Var}\left(\sum_{k=1}^{n} a_kX_kI(|a_kX_k| \le B_n)\right)$$

$$= \frac{1}{B_n^2}\sum_{k=1}^{n} a_k^2\operatorname{Var}(XI(|X| \le B_n/|a_k|))$$

$$= \frac{1}{B_n^2}\sum_{k=1}^{n} a_k^2EX^2I(|X| \le B_n/|a_k|) - \frac{1}{B_n^2}\sum_{k=1}^{n} a_k^2(EXI(|X| \le B_n/|a_k|))^2$$

$$= \frac{1}{B_n^2}\sum_{k=1}^{n} a_k^2H(B_n/|a_k|) - \frac{1}{B_n^2}\sum_{k=1}^{n} a_k^2(EXI(|X| \le B_n/|a_k|))^2 \to 1.$$

Next, noting that $X \in \mathcal{L}_1$ and

$$\max_{1\le k\le n} \frac{a_k^2(EX_k^{(n)})^2}{B_n^2} = \max_{1\le k\le n} \frac{a_k^2(EXI(|X| \le B_n/|a_k|))^2}{B_n^2}$$

$$\le \frac{\sum_{k=1}^{n} a_k^2(EXI(|X| \le B_n/|a_k|))^2}{B_n^2} \le \frac{s_n^2(E|X|)^2}{B_n^2}$$

$$= o(1) \qquad \text{(by (14))}$$

and hence

$$\max_{1\leq k\leq n} \frac{|a_k||EX_k^{(n)}|}{B_n} = o(1). \tag{18}$$

We can now establish that the normed sum of the truncated variables, $\frac{\sum_{k=1}^{n} a_k(X_k^{(n)}-EX_k^{(n)})}{\gamma_n}$, converges in distribution to a standard normal random variable.

Let $\varepsilon > 0$. Clearly $\gamma_n \sim B_n$ and (18) ensure that for all large n, $\max_{1\leq k\leq n} \left|\frac{a_k EX_k^{(n)}}{\gamma_n}\right| \leq \varepsilon/2$.

Hence for all large n and $1 \leq k \leq n$, $|EX_k^{(n)}| \leq \frac{\varepsilon\gamma_n}{2|a_k|}$ or, equivalently, $EX_k^{(n)} \leq \frac{\varepsilon\gamma_n}{2|a_k|}$ and $EX_k^{(n)} \geq \frac{-\varepsilon\gamma_n}{2|a_k|}$. Thus for all large n and all $1 \leq k \leq n$,

$$\{|X_k^{(n)}-EX_k^{(n)}| > \varepsilon\gamma_n/|a_k|\}$$

$$= \{X_k^{(n)}-EX_k^{(n)} < -\varepsilon\gamma_n/|a_k|\} \cup \{X_k^{(n)}-EX_k^{(n)} > \varepsilon\gamma_n/|a_k|\}$$

$$= \{X_k^{(n)} < EX_k^{(n)}-\varepsilon\gamma_n/|a_k|\} \cup \{X_k^{(n)} > EX_k^{(n)}+\varepsilon\gamma_n/|a_k|\}$$

$$\subset \{X_k^{(n)} < \frac{-\varepsilon\gamma_n}{2|a_k|}\} \cup \{X_k^{(n)} > \frac{\varepsilon\gamma_n}{2|a_k|}\}$$

$$= \{|X_k^{(n)}| > \frac{\varepsilon\gamma_n}{2|a_k|}\}.$$

Thus, for all $\varepsilon > 0$, and all large n,

$$\frac{1}{\gamma_n^2} \sum_{k=1}^{n} E(a_kX_k^{(n)}-Ea_kX_k^{(n)})^2 I(|a_kX_k^{(n)}-Ea_kX_k^{(n)}| > \varepsilon\gamma_n)$$

$$\leq \frac{2}{B_n^2} \sum_{k=1}^{n} a_k^2E(X_k^{(n)}-EX_k^{(n)})^2 I(|X_k^{(n)}-EX_k^{(n)}| > \varepsilon\gamma_n/|a_k|)$$

(since $\gamma_n \sim B_n$)

$$\leq \frac{2}{B_n^2} \sum_{k=1}^{n} a_k^2E(X_k^{(n)}-EX_k^{(n)})^2 I(|X_k^{(n)}| > \frac{\varepsilon\gamma_n}{2|a_k|})$$

(by the previous observation)

$$\leq \frac{4}{B_n^2} \sum_{k=1}^{n} a_k^2E((X_k^{(n)})^2 + (EX_k^{(n)})^2) I(|X_k^{(n)}| > \frac{\varepsilon\gamma_n}{2|a_k|})$$

(using the elementary inequality $(a-b)^2 \leq 2(a^2+b^2)$)

$$= \frac{4}{B_n^2} \sum_{k=1}^{n} a_k^2E(X_k^{(n)})^2 I(|X_k^{(n)}| > \frac{\varepsilon\gamma_n}{2|a_k|})$$

$$+ \frac{4}{B_n^2} \sum_{k=1}^{n} a_k^2(EX_k^{(n)})^2 P\{|X_k^{(n)}| > \frac{\varepsilon\gamma_n}{2|a_k|}\}.$$

This second sum is o(1) since it is dominated by $\frac{4}{B_n^2} s_n^2(E|X|)^2$, which is o(1), recalling $X \in \mathcal{L}_1$ and (14). The first sum is equal to

$$\frac{4}{B_n^2} \sum_{k=1}^{n} a_k^2EX_k^2 I(\frac{\varepsilon\gamma_n}{2|a_k|} < |X_k| \leq \frac{B_n}{|a_k|})$$

$$\leq \frac{4}{B_n^2} \sum_{k=1}^{n} B_n^2 P\left\{\frac{\varepsilon\gamma_n}{2|a_k|} < |X_k| \leq \frac{B_n}{|a_k|}\right\}$$

$$\leq 4 \sum_{k=1}^{n} P\left\{|X_k| > \frac{\varepsilon\gamma_n}{2|a_k|}\right\}$$

$$\leq 4 \sum_{k=1}^{n} P\left\{|X| > \frac{\varepsilon B_n}{4|a_k|}\right\} \qquad \text{(for all large } n \text{ since } \gamma_n \sim B_n\text{)}$$

$$= o(1) \qquad \text{(by (10))}.$$

Therefore by Theorem 2.3

$$\frac{S_n'-ES_n'}{\gamma_n} \xrightarrow{d} N(0,1). \tag{19}$$

Let $\delta > 0$. Then

$$P\left\{\frac{|S_n-S_n'|}{B_n} > \delta\right\}$$

$$\leq P\left\{\sum_{k=1}^{n} a_kX_k \neq \sum_{k=1}^{n} a_kX_k^{(n)}\right\}$$

$$\leq P\left\{\bigcup_{k=1}^{n} [|X_k| > B_n/|a_k|]\right\}$$

$$\leq \sum_{k=1}^{n} P\{|X_k| > B_n/|a_k|\} = o(1) \qquad \text{(by (10))}$$

and so

$$\frac{S_n-S_n'}{B_n} \xrightarrow{P} 0. \tag{20}$$

Hence

$$\frac{S_n - ES_n'}{B_n} = \frac{S_n - S_n' + S_n' - ES_n'}{B_n}$$

$$= \frac{S_n - S_n'}{B_n} + \left(\frac{\Upsilon_n}{B_n}\right)\left(\frac{S_n' - ES_n'}{\Upsilon_n}\right) \xrightarrow{d} N(0,1)$$

(by (19), (20), $B_n \sim \Upsilon_n$, and Slutsky's theorem).

Finally, noting that

$$ES_n' = E \sum_{k=1}^{n} a_k X_k^{(n)} = E \sum_{k=1}^{n} a_k X_k I(|X_k| \le B_n/|a_k|)$$

$$= \sum_{k=1}^{n} a_k EX_k I(|X_k| \le B_n/|a_k|) = A_n,$$

the conclusion (11) follows. □

With this result in hand, it is natural to ask, under what conditions is $A_n/B_n = o(1)$? It would seem natural to assume that $EX = 0$, but, that in addition to the other hypotheses of Theorem 2.4, does not seem to ensure that

$$\frac{\sum_{k=1}^{n} a_k X_k}{B_n} \xrightarrow{d} N(0,1). \tag{21}$$

It will be shown that we need to strengthen condition (8), and then, with a little help from Lemma 2.5, (21) will follow.

Lemma 2.6. If $P\{|X| > x\}$ is regularly varying with exponent -2, then G is slowly varying.

Proof. Without loss of generality assume that $EX^2 = \infty$. Let $\delta > 0$, $s > 0$. Since $P\{|X| > x\}$ is regularly varying with exponent -2,

$$1-\delta < \frac{s^2 P\{|X| > sx\}}{P\{|X| > x\}} < 1+\delta \text{ for all } x \geq \text{some } x_0. \tag{22}$$

Thus, for all $x \geq x_0$

$$G(sx) = \int_0^{sx} 2tP\{|X| > t\}dt$$

$$= \int_0^{sx_0} 2tP\{|X| > t\}dt + \int_{sx_0}^{sx} 2tP\{|X| > t\}dt$$

$$= G(sx_0) + 2\int_{x_0}^{x} ys^2 P\{|X| > ys\}dy$$

$$\leq G(sx_0) + (1+\delta)2\int_{x_0}^{x} yP\{|X| > y\}dy \qquad \text{(by (22))}$$

$$= G(sx_0) + (1+\delta)(G(x)-G(x_0))$$

implying that

$$\frac{G(sx)}{G(x)} \leq \frac{G(sx_0)}{G(x)} + (1+\delta)[1 - \frac{G(x_0)}{G(x)}].$$

Taking the limit superior of both sides as x → ∞ yields

$$\limsup_{x\to\infty} \frac{G(sx)}{G(x)} \leq 1+\delta. \tag{23}$$

Likewise, for all $x \geq x_o$

$$G(sx) = G(sx_o) + 2\int_{x_o}^{x} ys^2P\{|X| > ys\}dy$$

$$\geq G(sx_o) + (1-\delta)2\int_{x_o}^{x} yP\{|X| > y\}dy \qquad \text{(by (22))}$$

$$= G(sx_o) + (1-\delta)[G(x)-G(x_o)]$$

implying that

$$\frac{G(sx)}{G(x)} \geq \frac{G(sx_o)}{G(x)} + (1-\delta)[1 - \frac{G(x_o)}{G(x)}].$$

Taking limit inferiors as x → ∞ in this last inequality yields

$$\liminf_{x\to\infty} \frac{G(sx)}{G(x)} \geq 1-\delta.$$

Since δ is arbitrary, the desired result follows from this and (23).☐

The converse to Lemma 2.6 is false; for a counterexample see Feller (1971, p. 288).

At this point we are prepared to prove a CLT without the centering constants $\{A_n, n \geq 1\}$.

Corollary 2.1. If $\{X, X_n, n \geq 1\}$ are i.i.d. random variables with $EX = 0$, $EX^2 = \infty$, $P\{|X| > x\}$ regularly varying with exponent -2, and if (9) and (10) are satisfied, then (21) holds.

Proof. By Lemmas 2.6 and 2.2, condition (8) holds. In view of Theorem 2.4 it suffices to show that $A_n/B_n = o(1)$. Since $P\{|X| > x\}$ is regularly varying with exponent -2, we obtain via Lemma 2.5 with $\rho = -2$ that

$$E|X|I(|X| > x) \leq 3xP\{|X| > x\} \text{ for all } x \geq \text{some } x_o. \tag{24}$$

Recalling (15), we note that $B_n/\max_{1\leq k\leq n}|a_k| \to \infty$, whence there exists an integer n_o so that $B_n/\max_{1\leq k\leq n}|a_k| \geq x_o$ whenever $n \geq n_o$. So if $n \geq n_o$, we obtain $B_n/|a_k| \geq x_o$ for all $k = 1, \ldots, n$. Thus (24) implies

$$E|X|I(|X| > B_n/|a_k|) \leq 3\frac{B_n}{|a_k|}P\{|X| > B_n/|a_k|\} \text{ whenever } n \geq n_o. \tag{25}$$

Therefore when n is sufficiently large,

$$|A_n/B_n| = \frac{1}{B_n}\left|\sum_{k=1}^{n} a_k EXI(|X| \leq B_n/|a_k|)\right|$$

$$= \frac{1}{B_n}\left|-\sum_{k=1}^{n} a_k EXI(|X| > B_n/|a_k|)\right| \qquad \text{(since } EX = 0\text{)}$$

$$\leq \frac{1}{B_n}\sum_{k=1}^{n} |a_k|E|X|I(|X| > B_n/|a_k|)$$

$$\leq \frac{1}{B_n} \sum_{k=1}^{n} |a_k| \frac{3B_n}{|a_k|} P\{|X| > B_n/|a_k|\} \qquad \text{(by (25))}$$

$$= 3 \sum_{k=1}^{n} P\{|X| > B_n/|a_k|\} = o(1) \qquad \text{(by (10))}. \ \square$$

The following two corollaries will be shown to be immediate consequences of Theorem 2.4.

Corollary 2.2. Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables with $EX^2 = \infty$ and satisfying (8). Let $\{a_n, n \geq 1\}$ be constants so that

$$n \max_{1\leq k\leq n} a_k^2 = O(s_n^2) \tag{26}$$

and

$$G\left(\frac{B_n}{\min\limits_{1\leq k\leq n} |a_k|}\right) \sim G(q_n). \tag{27}$$

Then (11) holds.

Proof. In view of Theorem 2.4, we only need to verify that (9) and (10) prevail.

Let $\alpha_n = \dfrac{B_n}{\max\limits_{1\leq k\leq n} |a_k|}$, $n \geq 1$, and observe that $\alpha_n \geq \dfrac{B_n}{s_n} \to \infty$. By (26), there is some constant $C < \infty$ such that $\dfrac{n \max\limits_{1\leq k\leq n} a_k^2}{s_n^2} \leq C^2$, $n \geq 1$.

Then

$$C^2 \geq \frac{n \max\limits_{1\leq k\leq n} a_k^2}{s_n^2} = \left(\frac{\max\limits_{1\leq k\leq n} a_k^2}{B_n^2}\right)\left(\frac{nB_n^2}{s_n^2}\right) = \frac{q_n^2}{\alpha_n^2} \qquad \text{(by (7))},$$

and clearly $\dfrac{s_n}{\sqrt{n}\max\limits_{1\leq k\leq n}|a_k|} \leq 1$ implying that

$$\alpha_n \leq q_n \leq C\alpha_n,\ n \geq 1. \tag{28}$$

For arbitrary $\varepsilon > 0$, note that

$$\frac{nH(\frac{\varepsilon}{C}q_n)}{q_n^2} \sim \frac{nH(q_n)}{q_n^2} \qquad \text{(since (8) implies that H is slowly varying)}$$

$$\sim \frac{nG(q_n)}{q_n^2} \qquad \text{(since } H(x) \sim G(x) \text{ via (8))}$$

$$= 1 \qquad \text{(by (13))}.$$

Thus

$$nH(\tfrac{\varepsilon}{C}q_n) \sim q_n^2. \tag{29}$$

Then, for all arbitrary $\varepsilon > 0$ and n sufficiently large

$$\sum_{k=1}^{n} P\{|X| > \varepsilon B_n/|a_k|\}$$

$$\leq \sum_{k=1}^{n} P\{|X| > \varepsilon B_n/\max_{1\leq j\leq n}|a_j|\}$$

$$= nP\{|X| > \varepsilon\alpha_n\}$$

$$\le nP\{|X| > \tfrac{\varepsilon}{C}q_n\} \qquad \text{(by (28))}$$

$$= \frac{n(\frac{\varepsilon}{C}q_n)^2}{(\frac{\varepsilon}{C}\, q_n)^2} \cdot P\{|X| > \tfrac{\varepsilon}{C}q_n\} \cdot \frac{H(\frac{\varepsilon}{C}q_n)}{H(\frac{\varepsilon}{C}q_n)}$$

$$= (\tfrac{C}{\varepsilon})^2 \cdot \frac{(\frac{\varepsilon}{C}q_n)^2 P\{|X| > \frac{\varepsilon}{C}q_n\}}{H(\frac{\varepsilon}{C}q_n)} \cdot \frac{nH(\frac{\varepsilon}{C}q_n)}{q_n^2}$$

$$= (1+o(1))(\tfrac{C}{\varepsilon})^2 \frac{(\frac{\varepsilon}{C}q_n)^2 P\{|X| > \frac{\varepsilon}{C}q_n\}}{H(\frac{\varepsilon}{C}q_n)} \qquad \text{(by (29))}$$

$$= o(1) \qquad \text{(by (8))}$$

thereby proving (10).

To prove (9), in view of (27), we need only show that

$$G(q_n) \sim G\left(\frac{B_n}{\max\limits_{1\le k\le n} |a_k|}\right). \tag{30}$$

Note that (28) entails

$$\frac{1}{C}q_n \le \frac{B_n}{\max\limits_{1\le k\le n} |a_k|} \le q_n.$$

Utilizing the fact that G is nondecreasing we obtain

$$G(\frac{1}{C}q_n) \leq G(\frac{B_n}{\max\limits_{1\leq k\leq n} |a_k|}) \leq G(q_n).$$

Thus (30) holds since G is slowly varying. $\square$

Corollary 2.3. Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables with $EX^2 = \infty$ and satisfying (8). Let $\{a_n, n \geq 1\}$ be constants so that

$$0 < \inf_{n\geq 1} a_n^2 \leq \sup_{n\geq 1} a_n^2 < \infty \tag{31}$$

then

$$\frac{\sum\limits_{k=1}^{n} a_k X_k - A_n}{B_n} \xrightarrow{d} N(0,1)$$

where $\{A_n, n \geq 1\}$ is defined as in Theorem 2.4.

Proof. Again we will verify that (9) and (10) are satisfied.

Clearly,

$$\frac{B_n}{\sup\limits_{k\geq 1} |a_k|} \leq \frac{B_n}{\max\limits_{1\leq k\leq n} |a_k|} \leq \frac{B_n}{\min\limits_{1\leq k\leq n} |a_k|} \leq \frac{B_n}{\inf\limits_{k\geq 1} |a_k|}$$

which, since G is nondecreasing, implies that

$$G\left(\frac{B_n}{\sup\limits_{k\geq 1}|a_k|}\right) \leq G\left(\frac{B_n}{\max\limits_{1\leq k\leq n}|a_k|}\right) \leq G\left(\frac{B_n}{\min\limits_{1\leq k\leq n}|a_k|}\right) \leq G\left(\frac{B_n}{\inf\limits_{k\geq 1}|a_k|}\right), \; n \geq 1.$$

Recalling that (8) is equivalent to G slowly varying we obtain

$$G\left(\frac{B_n}{\sup\limits_{k\geq 1}|a_k|}\right) \sim G\left(\frac{B_n}{\inf\limits_{k\geq 1}|a_k|}\right)$$

by employing (31). Hence (9) obtains.

Define $\alpha_n = B_n/\max\limits_{1\leq k\leq n}|a_k|$ and $\beta_n = B_n/\min\limits_{1\leq k\leq n}|a_k|$, $n \geq 1$, and note that $B_n \geq \alpha_n \geq \frac{B_n}{s_n} \to \infty$. In light of (31), there exists constants $0 < C_1 \leq C_2 < \infty$ such that

$$C_1 \leq \min_{1\leq k\leq n}|a_k| \leq \max_{1\leq k\leq n}|a_k| \leq C_2, \text{ for all } n \geq 1.$$

Recall that $q_n^2 = nB_n^2/s_n^2$ and thus

$$\frac{B_n}{C_2} \leq \alpha_n \leq q_n \leq \beta_n \leq \frac{B_n}{C_1}, \; n \geq 1. \tag{32}$$

Again using the fact that G is nondecreasing we obtain

$$G\left(\frac{B_n}{C_2}\right) \leq G(\alpha_n) \leq G(q_n) \leq G\left(\frac{B_n}{C_1}\right).$$

This together with G slowly varying, ensures that for all $\varepsilon > 0$

$$H(\varepsilon\alpha_n) \sim G(\varepsilon\alpha_n) \sim G(\alpha_n) \sim G(q_n) \tag{33}$$

From (32) it follows that

$$q_n \leq \frac{B_n}{C_1} \leq \frac{C_2}{C_1}\alpha_n$$

or

$$\frac{1}{\alpha_n^2} \leq \left(\frac{C_2}{C_1}\right)^2 \frac{1}{q_n^2}. \tag{34}$$

Thus, for all $\varepsilon > 0$,

$$\begin{aligned}\frac{nH(\varepsilon\alpha_n)}{\alpha_n^2} &= \frac{(1+o(1))nG(q_n)}{\alpha_n^2} && \text{(by (33))}\\ &\leq (1+o(1))\left(\frac{C_2}{C_1}\right)^2 \frac{nG(q_n)}{q_n^2} && \text{(by (34))}\\ &= (1+o(1))\left(\frac{C_2}{C_1}\right)^2 && \text{(by (13))}\end{aligned}$$

whence

$$\frac{nH(\varepsilon\alpha_n)}{\alpha_n^2} = O(1). \tag{35}$$

Therefore for arbitrary $\varepsilon > 0$,

$$\begin{aligned}\sum_{k=1}^{n} P\{|X| > \varepsilon B_n/|a_k|\} &\leq nP\{|X| > \varepsilon\alpha_n\}\\ &= \frac{1}{\varepsilon^2} \cdot \frac{(\varepsilon\alpha_n)^2 P\{|X| > \varepsilon\alpha_n\}}{H(\varepsilon\alpha_n)} \cdot \frac{nH(\varepsilon\alpha_n)}{\alpha_n^2}\\ &= o(1) && \text{(by (8) and (35))}\end{aligned}$$

thereby establishing (10). □

Theorem 2.4 does have a partial converse. We, however, impose a relatively strong condition about the weights $\{a_n, n \geq 1\}$.

Remark. Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables with $EX^2 = \infty$. Let $\{a_n, n \geq 1\}$ be a sequence of constants where

$$s_n^2 = O(n \min_{1 \leq k \leq n} a_k^2). \tag{36}$$

Then (11) implies (8) and (10).

Proof. By (36) there exists a constant $0 < C < \infty$, such that $\dfrac{s_n^2}{n \min_{1 \leq k \leq n} a_k^2} \leq C$, $n \geq 1$, which implies that

$$\frac{1}{C} \leq \frac{n \min_{1 \leq k \leq n} a_k^2}{s_n^2}, \text{ for all } n \geq 1. \tag{37}$$

Note that, by definition of H, $H(x)/x^2 = o(1)$ as $x \to \infty$. Then, recalling (6), $G(x)/x^2 = o(1)$, whence $Q(x) = x^2/G(x) \to \infty$ as $x \to \infty$. This, in turn, shows that $q_n \to \infty$. Therefore $B_n^2 = s_n^2 G(q_n) \to \infty$ and

$$\frac{B_n}{\min_{1 \leq k \leq n} |a_k|} \geq \frac{B_n}{\max_{1 \leq k \leq n} |a_k|} \geq \frac{B_n}{s_n} \to \infty. \tag{38}$$

If we can show that

$$\max_{1 \leq k \leq n} P\{|X| > \delta B_n/|a_k|\} = o(1) \text{ for all } \delta > 0, \tag{39}$$

then, employing Corollary 12.2.3 of Chow and Teicher (1978), we can conclude that (10) holds and

$$\frac{1}{B_n^2}\sum_{k=1}^{n} a_k^2[H(\varepsilon B_n/|a_k|) - (EXI(|X| \le \varepsilon B_n/|a_k|))^2] \to 1 \text{ for all } \varepsilon > 0. \tag{40}$$

So to establish (39), note that for arbitrary $\delta > 0$

$$\max_{1\le k\le n} P\{|X| > \delta B_n/|a_k|\}$$

$$= P\{|X| > \delta B_n/ \max_{1\le k\le n} |a_k|\}$$

$$\le P\{|X| > \delta B_n/s_n\}$$

$$= o(1) \qquad \text{(by (38))}$$

whence (39) obtains. Then using (10) and (40), we conclude that for $\varepsilon = 1$

$$o(1) = \frac{\sum_{k=1}^{n} P\{|X| > B_n/|a_k|\}}{\frac{1}{B_n^2}\sum_{k=1}^{n} a_k^2[H(B_n/|a_k|) - (EXI(|X| \le B_n/|a_k|))^2]}$$

$$\ge \frac{B_n^2 \sum_{k=1}^{n} P\{|X| > B_n/|a_k|\}}{\sum_{k=1}^{n} a_k^2 H(B_n/|a_k|)}$$

$$\geq \frac{B_n^2 \sum_{k=1}^{n} P\{|X| > B_n / \min_{1 \leq k \leq n} |a_k|\}}{\sum_{k=1}^{n} a_k^2 H(B_n / \min_{1 \leq k \leq n} |a_k|)}$$

$$= \frac{nB_n^2 P\{|X| > B_n / \min_{1 \leq k \leq n} |a_k|\}}{s_n^2 H(B_n / \min_{1 \leq k \leq n} |a_k|)}$$

$$= \frac{n \min_{1 \leq k \leq n} a_k^2}{s_n^2} \cdot \left(\frac{B_n}{\min_{1 \leq k \leq n} |a_k|}\right)^2 \cdot \frac{P\{|X| > B_n / \min_{1 \leq k \leq n} |a_k|\}}{H(B_n / \min_{1 \leq k \leq n} |a_k|)}$$

$$\geq \frac{1}{C}\left(\frac{B_n}{\min_{1 \leq k \leq n} |a_k|}\right)^2 \cdot \frac{P\{|X| > B_n / \min_{1 \leq k \leq n} |a_k|\}}{H(B_n / \min_{1 \leq k \leq n} |a_k|)} \qquad \text{(by (37)).}$$

Thus

$$\left(\frac{B_n}{\min_{1 \leq k \leq n} |a_k|}\right)^2 \cdot \frac{P\{|X| > B_n / \min_{1 \leq k \leq n} |a_k|\}}{H(B_n / \min_{1 \leq k \leq n} |a_k|)} = o(1).$$

Then for all x such that

$$\frac{B_n}{\min_{1 \leq k \leq n} |a_k|} \leq x \leq \frac{2B_n}{\min_{1 \leq k \leq n} |a_k|}$$

we observe that

$$\frac{x^2P\{|X| > x\}}{H(x)} \le \left(\frac{2B_n}{\min\limits_{1\le k\le n}|a_k|}\right)^2 \cdot \frac{P\{|X| > \frac{B_n}{\min\limits_{1\le k\le n}|a_k|}\}}{H(B_n/\min\limits_{1\le k\le n}|a_k|)} \qquad \text{(since } H(x)\uparrow)$$

$$= 4\left(\frac{B_n}{\min\limits_{1\le k\le n}|a_k|}\right)^2 \cdot \frac{P\{|X| > B_n/\min\limits_{1\le k\le n}|a_k|\}}{H(B_n/\min\limits_{1\le k\le n}|a_k|)} = o(1) \quad \text{as} \quad x \to \infty$$

recalling (38) thereby establishing (8). □

The next corollary is an immediate consequence of Theorem 2.4 and this last remark. It is the famous result of Lévy (1935), Khintchine (1935), and Feller (1935) cited previously as Theorem 2.2.

Corollary 2.4 (Chow and Teicher, 1978, p. 300). If $\{X, X_n, n \ge 1\}$ are i.i.d. random variables with $EX^2 = \infty$, then

$$\frac{\sum\limits_{k=1}^{n} X_k - A_n^*}{B_n} \xrightarrow{d} N(0,1)$$

for some $B_n > 0$ and A_n^* iff (8) holds; moreover, B_n may be chosen as $q_n = Q^{-1}(n)$, $n \ge 1$, while A_n^* may be taken as

$$A_n^* = nEXI(|X| \le B_n), \; n \ge 1.$$

Proof. Clearly, $0 < \inf_{n \geq 1} a_n^2 \leq \sup_{n \geq 1} a_n^2 < \infty$, since $a_n = 1$, $n \geq 1$. Also

$$A_n = \sum_{k=1}^{n} a_k EXI(|X| \leq B_n/|a_k|)$$

$$= nEXI(|X| \leq B_n) = A_n^*, \ n \geq 1,$$

whence the sufficiency portion of the corollary follows from Corollary 2.3. Necessity follows from the last remark since $s_n^2 = n \min_{1 \leq k \leq n} a_k^2$. □

2.4 A Properly Centered Central Limit Theorem

We have seen that (8), via Lemma 2.3, implies that $X \in \mathcal{X}_1$. Then, if we assume $EX = 0$, $EX^2 = \infty$, (9), (10), and $P\{|X| > x\}$ regularly varying with exponent -2 we showed in Corollary 2.1 that (21) holds. It is natural to ask what happens when the mean of X is finite but not zero. Can we just shift the variables by EX and achieve asymptotic normality? In other words, when does the CLT

$$\frac{\sum_{k=1}^{n} a_k(X_k - EX)}{B_n} \xrightarrow{d} N(0,1) \tag{41}$$

hold?

At this point we need to introduce some new notation. Consider two sequences of random variables $\{X, X_n, n \geq 1\}$ and $\{Y, Y_n, n \geq 1\}$.

The $\{X, X_n, n \geq 1\}$ are i.i.d. with arbitrary finite mean, while the sequence $\{Y, Y_n, n \geq 1\}$ are defined by $Y = X-EX$ and $Y_n = X_n-EX$, $n \geq 1$. Again we suppose that $EX^2 = \infty$ and thus $EY^2 = \infty$.

The functions H and G will be defined as in (4) and (5). Likewise $s_n^2 = \sum_{k=1}^{n} a_k^2$, $n \geq 1$, $Q(x) = x^2/G(x)$, $q_n = Q^{-1}(n)$, $n \geq 1$, and $B_n = s_n q_n/\sqrt{n}$, $n \geq 1$. Similarly we need to define analogous quantities in terms of the random variable Y.

Let

$$H_1(y) = EY^2 I(|Y| \leq y),\ y \geq 0 \tag{42}$$

and

$$G_1(y) = \int_0^y 2tP\{|Y| > t\}dt,\ y \geq 0. \tag{43}$$

Also let $Q_1(y) = y^2/G_1(y)$, $q_n^* = Q_1^{-1}(n)$, $n \geq 1$, and

$$B_n^* = s_n q_n^*/\sqrt{n},\ n \geq 1. \tag{44}$$

Before we establish the relationship between these pairs of functions and sequences we need a few preliminary lemmas.

Lemma 2.7 (Rosalsky, 1981). If $\ell(y)$ is the inverse of the continuous increasing function $y^r L(y)$, $y > 0$ where L is slowly varying and $r > 0$, then $y^{-\frac{1}{r}}\ell(y)$ is slowly varying.

Proof. See Rosalsky (1981), Lemma 6. ☐

Lemma 2.8. Suppose that H is slowly varying and $EX^2 = \infty$. Then $H_1(t) \sim H(t)$ as $t \to \infty$ and H_1 is slowly varying.

Proof. Let $\mu = EX$ and recall that $Y = X-\mu$. Then, recalling (42), for all $t \geq 0$

$$\begin{aligned} H_1(t) &= EY^2I(|Y| \leq t) \\ &= E(X-\mu)^2I(|X-\mu| \leq t) \\ &= EX^2I(|X-\mu| \leq t) - 2\mu EXI(|X-\mu| \leq t) + \mu^2EI(|X-\mu| \leq t). \end{aligned}$$

Now for $t \geq 0$, since $X \in \mathcal{L}_1$ via Lemma 2.3

$$|2\mu EXI(|X-\mu| \leq t)| \leq 2|\mu|E|X| < \infty.$$

Also for $t \geq 0$

$$\mu^2EI(|X-\mu| \leq t) = \mu^2P\{|X-\mu| \leq t\} \leq \mu^2 < \infty.$$

Therefore, since $H(t) \to \infty$ as $t \to \infty$ (from $EX^2 = \infty$) we obtain

$$\frac{H_1(t)}{H(t)} = \frac{EX^2I(|X-\mu| \leq t)}{H(t)} + o(1). \tag{45}$$

For $t > |\mu|$,

$$EX^2I(|X-\mu| \leq t) = EX^2I(\mu-t \leq X \leq \mu+t)$$

$$\leq EX^2I(-2t \leq X \leq 2t)$$

$$= EX^2I(|X| \leq 2t) = H(2t)$$

Also, for $t \geq 2|\mu|$,

$$EX^2I(|X-\mu| \leq t)$$

$$= EX^2I(\mu-t \leq X \leq \mu+t)$$

$$\geq EX^2(-\tfrac{1}{2}t \leq X \leq \tfrac{1}{2}t)$$

$$= EX^2I(|X| \leq \tfrac{1}{2}t) = H(\tfrac{1}{2}t).$$

These two results entail, for sufficiently large t

$$\frac{H(\frac{1}{2}t)}{H(t)} \leq \frac{EX^2I(|X-\mu| \leq t)}{H(t)} \leq \frac{H(2t)}{H(t)},$$

and since H is slowly varying we obtain $EX^2I(|X-\mu| \leq t) \sim H(t)$. In view of (45) this yields $H_1(t) \sim H(t)$ as $t \to \infty$.

Finally, for $s > 0$,

$$\begin{aligned} H_1(st) &\sim H(st) && \text{(since } H_1(t) \sim H(t)\text{)} \\ &\sim H(t) && \text{(since } H \text{ is slowly varying)} \\ &\sim H_1(t) && \text{(since } H_1(t) \sim H(t)\text{)}, \end{aligned}$$

And so H_1 is slowly varying. □

With this relationship between H_1 and H established, we note that a similar relationship clearly must then exist between G_1 and G when H is slowly varying in view of Lemma 2.2. The natural question is whether $B_n^* \sim B_n$. Prior to establishing this we need to state two facts about functions of slow variation. The following result (see, e.g., Feller, 1971, p. 282) characterizes the class of slowing varying functions and is known as the Karamata representation theorem.

Theorem. A function L(t) varies slowly iff it is of the form

$$L(t) = a(t)\exp\left\{\int_1^t \frac{\varepsilon(s)}{s}ds\right\} \tag{46}$$

where $\varepsilon(t) \to 0$ and $a(t) \to c$, $0 < c < \infty$, as $t \to \infty$.

This next lemma will demonstrate the value of the Karamata representation theorem and is a quite useful result.

Lemma 2.9. If $0 < u(t) \sim v(t) \to \infty$ as $t \to \infty$ and L is slowly varying, then

$$L(u(t)) \sim L(v(t)).$$

Proof. Since L(t) is slowly varying, by (46), $L(t) = a(t)\exp\{\int_1^t \frac{\varepsilon(y)}{y}dy\}$, where $\varepsilon(t) \to 0$ and $a(t) \to c$, $0 < c < \infty$, as $t \to \infty$.

Thus,

$$\frac{L(u(t))}{L(v(t))} = \frac{a(u(t))}{a(v(t))}\exp\{\int_1^{u(t)} \frac{\varepsilon(y)}{y}dy - \int_1^{v(t)} \frac{\varepsilon(y)}{y}dy\}$$

$$\leq (1+o(1))\exp\{\int_{\min\{u(t),v(t)\}}^{\max\{u(t),v(t)\}} \frac{|\varepsilon(y)|}{y}dy\}, \tag{47}$$

since $a(x) \to c$ as $x \to \infty$ and $u(t), v(t) \to \infty$ as $t \to \infty$.

Using $u(t) \sim v(t) \to \infty$ and $\varepsilon(y) \to 0$ we observe for all t sufficiently large that $|\varepsilon(y)| \leq 1$ for all $y \geq \min\{u(t),v(t)\}$. Hence, via (47), for all large t

$$\frac{L(u(t))}{L(v(t))} \leq (1+o(1))\exp\{\int_{\min\{u(t),v(t)\}}^{\max\{u(t),v(t)\}} \frac{1}{y}dy\}$$

$$= (1+o(1))\frac{\max\{u(t),v(t)\}}{\min\{u(t),v(t)\}}$$

$$= (1+o(1))\max\{\frac{u(t)}{v(t)}, \frac{v(t)}{u(t)}\} = 1+o(1). \tag{48}$$

Reversing the roles of u and v we obtain for all large t

$$\frac{L(v(t))}{L(u(t))} \leq 1+o(1). \tag{49}$$

Combining (48) and (49) we conclude that $L(u(t)) \sim L(v(t))$. □

Lemma 2.10. If $EX^2 = \infty$ and H is slowly varying, then $B_n^* \sim B_n$.

Proof. Recall (42) and (43). By Lemmas 2.2 and 2.8, $G(t) \sim H(t) \sim H_1(t) \sim G_1(t)$ whence $Q(t) = t^2/G(t) \sim t^2/G_1(t) = Q_1(t)$.

Let $q(t) = Q^{-1}(t)$ and $q_1(t) = Q_1^{-1}(t)$, $t > 0$. Then $Q_1(q_1(t)) = t = Q(q(t)) \sim Q_1(q(t))$ (since $Q(t) \sim Q_1(t)$), and so

$$Q_1(q_1(t)) \sim Q_1(q(t)). \tag{50}$$

By Lemma 2.7, $L(t) \equiv t^{-1/2}q_1(t)$ is slowly varying since $Q_1(t) = t^2/G_1(t)$ is increasing (see Rosalsky, 1981, Lemma 2) and G_1 is slowly varying. Next, note that if $u(t) \sim v(t) \to \infty$ then, via Lemma 2.9, $L(u(t)) \sim L(v(t))$. Applying this to (50) we see that

$$L(Q_1(q_1(t))) \sim L(Q_1(q(t)))$$

or

$$[Q_1(q_1(t))]^{-1/2} \cdot q_1(Q_1(q_1(t))) \sim [Q_1(q(t))]^{-1/2} \cdot q_1(Q_1(q(t)))$$

which, via (50), is tantamount to

$$q_1(Q_1(q_1(t))) \sim q_1(Q_1(q(t))).$$

Using $q_1(t) = Q_1^{-1}(t)$ we obtain $q_1(t) \sim q(t)$ or $q_n^* \sim q_n$. Therefore, recalling (44) and (7),

$$B_n^* = s_n q_n^*/\sqrt{n} \sim s_n q_n/\sqrt{n} = B_n. \quad \square$$

Lemma 2.11. If $EX^2 = \infty$ and H is slowly varying, then

$$G\left(\frac{B_n}{\min\limits_{1\le k\le n} |a_k|}\right) \sim G\left(\frac{B_n}{\max\limits_{1\le k\le n} |a_k|}\right)$$

iff

$$G_1\left(\frac{B_n^*}{\min\limits_{1\le k\le n} |a_k|}\right) \sim G_1\left(\frac{B_n^*}{\max\limits_{1\le k\le n} |a_k|}\right).$$

Proof. From Lemma 2.10 we have $B_n^* \sim B_n$. Now H slowly varying implies, via Lemma 2.2, that G is slowly varying. Hence by $\dfrac{B_n}{\min\limits_{1\le k\le n} |a_k|} \sim \dfrac{B_n^*}{\min\limits_{1\le k\le n} |a_k|} \to \infty$ we obtain

$$G\left(\frac{B_n}{\min\limits_{1\le k\le n} |a_k|}\right) \sim G\left(\frac{B_n^*}{\min\limits_{1\le k\le n} |a_k|}\right) \qquad \text{(by Lemma 2.9)}$$

$$\sim G_1\left(\frac{B_n^*}{\min\limits_{1\le k\le n}|a_k|}\right) \qquad \text{(by Lemmas 2.8 and 2.2).} \tag{51}$$

Likewise

$$G\left(\frac{B_n}{\max\limits_{1\le k\le n}|a_k|}\right) \sim G\left(\frac{B_n^*}{\max\limits_{1\le k\le n}|a_k|}\right) \sim G_1\left(\frac{B_n^*}{\max\limits_{1\le k\le n}|a_k|}\right). \tag{52}$$

Thus, if $G\left(\frac{B_n}{\min\limits_{1\le k\le n}|a_k|}\right) \sim G\left(\frac{B_n}{\max\limits_{1\le k\le n}|a_k|}\right)$, then

$$G_1\left(\frac{B_n^*}{\min\limits_{1\le k\le n}|a_k|}\right) \sim G\left(\frac{B_n}{\min\limits_{1\le k\le n}|a_k|}\right) \qquad \text{(by (51))}$$

$$\sim G\left(\frac{B_n}{\max\limits_{1\le k\le n}|a_k|}\right) \sim G_1\left(\frac{B_n^*}{\max\limits_{1\le k\le n}|a_k|}\right) \qquad \text{(by (52)).}$$

Conversely, if $G_1\left(\frac{B_n^*}{\min\limits_{1\le k\le n}|a_k|}\right) \sim G_1\left(\frac{B_n^*}{\max\limits_{1\le k\le n}|a_k|}\right)$, then

$$G\left(\frac{B_n}{\min\limits_{1\le k\le n}|a_k|}\right) \sim G_1\left(\frac{B_n^*}{\min\limits_{1\le k\le n}|a_k|}\right) \qquad \text{(by (51))}$$

$$\sim G_1\left(\frac{B_n^*}{\max\limits_{1\le k\le n}|a_k|}\right) \sim G\left(\frac{B_n}{\max\limits_{1\le k\le n}|a_k|}\right) \qquad \text{(by (52)).} \quad \square$$

Lemma 2.12. If $EX^2 = \infty$, H is slowly varying, and

$$\sum_{k=1}^{n} P\{|X| > \varepsilon B_n/|a_k|\} = o(1) \qquad \text{for all } \varepsilon > 0,$$

then

$$\sum_{k=1}^{n} P\{|Y| > \varepsilon B_n^*/|a_k|\} = o(1) \qquad \text{for all } \varepsilon > 0.$$

Proof. Let $\varepsilon > 0$. By Lemma 2.10, there exists a constant $M < \infty$ such that

$$B_n/B_n^* \leq M \qquad \text{for all } n \geq 1. \tag{53}$$

Recall that $B_n/\max_{1\leq k\leq n} |a_k| \to \infty$ and so if n is sufficiently large, then $\frac{\varepsilon B_n}{2M|a_k|} \geq |\mu|$ for all $k = 1, \ldots, n$ where $\mu = EX$. Therefore

$$\sum_{k=1}^{n} P\{|Y| > \varepsilon B_n^*/|a_k|\}$$

$$\leq \sum_{k=1}^{n} P\{|X-\mu| > \frac{\varepsilon B_n}{M|a_k|}\} \qquad \text{(by (53))}$$

$$= \sum_{k=1}^{n} [P\{X < \mu - \frac{\varepsilon B_n}{M|a_k|}\} + P\{X > \mu + \frac{\varepsilon B_n}{M|a_k|}\}]$$

$$\leq \sum_{k=1}^{n} [P\{X < \frac{-\varepsilon B_n}{2M|a_k|}\} + P\{X > \frac{\varepsilon B_n}{2M|a_k|}\}]$$

$$= \sum_{k=1}^{n} P\{|X| > \frac{\varepsilon B_n}{2M|a_k|}\} = o(1) \qquad \text{(by hypothesis).} \ \square$$

Lemma 2.13. If $P\{|X| > t\}$ is regularly varying with exponent -2, then $P\{|Y| > t\}$ is likewise regularly varying with exponent -2.

Proof. Let $L(t) = t^2P\{|X| > t\}$ and $L_1(t) = t^2P\{|Y| > t\}$, for $t \geq 0$. It need only be shown that $L_1(t) \sim L(t)$ since then, for arbitrary $s > 0$,

$$L_1(st) \sim L(st) \sim L(t) \sim L_1(t).$$

To this end, let $0 < \varepsilon < 1$ be arbitrary and let $t > (1+\varepsilon)|\mu|/\varepsilon$ where $\mu = EX$. Then

$$\frac{t}{1+\varepsilon} < \mu+t \quad \text{and} \quad \frac{-t}{1+\varepsilon} > \mu-t$$

whence

$$\frac{L_1(t)}{L(t)} = \frac{t^2P\{|Y| > t\}}{t^2P\{|X| > t\}} = \frac{P\{|X-\mu| > t\}}{P\{|X| > t\}}$$

$$= \frac{P\{X < \mu-t\} + P\{X > \mu+t\}}{P\{|X| > t\}}$$

$$\leq \frac{P\{X < \frac{-t}{1+\varepsilon}\} + P\{X > \frac{t}{1+\varepsilon}\}}{P\{|X| > t\}}$$

$$= \frac{P\{|X| > \frac{t}{1+\varepsilon}\}}{P\{|X| > t\}} = \frac{(1+\varepsilon)^2(\frac{t}{1+\varepsilon})^2P\{|X| > \frac{t}{1+\varepsilon}\}}{t^2P\{|X| > t\}}$$

$$= \frac{(1+\varepsilon)^2 L(\frac{t}{1+\varepsilon})}{L(t)} = (1+\varepsilon)^2(1+o(1))$$

since, by hypothesis, L is slowly varying.

Hence $\limsup_{t\to\infty} \frac{L_1(t)}{L(t)} \le (1+\varepsilon)^2$, and since ε is arbitrary we obtain

$$\limsup_{t\to\infty} \frac{L_1(t)}{L(t)} \le 1. \tag{54}$$

Again, let $0 < \varepsilon < 1$ be arbitrary, and now let $t > (1-\varepsilon)|\mu|/\varepsilon$, which implies that

$$\frac{t}{1-\varepsilon} > \mu+t \quad \text{and} \quad \frac{-t}{1-\varepsilon} < \mu-t.$$

Therefore

$$\frac{L_1(t)}{L(t)} = \frac{P\{|X-\mu| > t\}}{P\{|X| > t\}}$$

$$= \frac{P\{X < \mu-t\} + P\{X > \mu+t\}}{P\{|X| > t\}}$$

$$\ge \frac{P\{X < \frac{-t}{1-\varepsilon}\} + P\{X > \frac{t}{1-\varepsilon}\}}{P\{|X| > t\}}$$

$$= \frac{P\{|X| > \frac{t}{1-\varepsilon}\}}{P\{|X| > t\}} = \frac{(1-\varepsilon)^2(\frac{t}{1-\varepsilon})^2 P\{|X| > \frac{t}{1-\varepsilon}\}}{t^2 P\{|X| > t\}}$$

$$= \frac{(1-\varepsilon)^2 L(\frac{t}{1-\varepsilon})}{L(t)} = (1-\varepsilon)^2(1+o(1))$$

since L is slowly varying. Hence $\liminf_{t\to\infty} \frac{L_1(t)}{L(t)} \geq (1-\varepsilon)^2$, whence

$$\liminf_{t\to\infty} \frac{L_1(t)}{L(t)} \geq 1.$$

This, together with (54), shows that $L_1(t) \sim L(t)$ which is tantamount to $P\{|Y| > t\}$ regularly varying with exponent -2. □

We are now able to state and prove a CLT for random variables centered about their mean.

Theorem 2.5. Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables with $EX^2 = \infty$ and let $P\{|X| > t\}$ be regularly varying with exponent -2. Let $\{a_n, n \geq 1\}$ be constants with

$$G\left(\frac{B_n}{\min_{1\leq k\leq n} |a_k|}\right) \sim G\left(\frac{B_n}{\max_{1\leq k\leq n} |a_k|}\right) \tag{55}$$

and

$$\sum_{k=1}^{n} P\{|X| > \varepsilon B_n/|a_k|\} = o(1) \quad \text{for all } \varepsilon > 0. \tag{56}$$

Then

$$\frac{\sum_{k=1}^{n} a_k(X_k - EX)}{B_n} \xrightarrow{d} N(0,1).$$

Proof. Via Lemma 2.6 and regular variation of $P\{|X| > t\}$, we can conclude that G and H are slowly varying. This, in turn, implies that $X \in \mathcal{L}_1$ by Lemma 2.3. Let $Y = X-EX$, $Y_n = X_n-EX$, $n \geq 1$, and note that since $EX^2 = \infty$ we also have $EY^2 = \infty$.

Now by Lemma 2.13 and the hypothesis that $P\{|X| > t\}$ is regularly varying with exponent -2, we see that $P\{|Y| > t\}$ is also regularly varying with exponent -2. Again by Lemma 2.6 we can conclude that G_1 and H_1 are also slowly varying. Next, via Lemma 2.10, we conclude that $B_n^* \sim B_n$. Now (55) is equivalent to $G_1(B_n^*/ \min_{1\leq k\leq n} |a_k|) \sim G_1(B_n^*/ \max_{1\leq k\leq n} |a_k|)$ by Lemma 2.11. Finally we note by Lemma 2.12 that (56) implies $\sum_{k=1}^{n} P\{|Y| > \varepsilon B_n^*/|a_k|\} = o(1)$ for all $\varepsilon > 0$. Therefore, all the hypotheses of Corollary 2.1 are satisfied for the sequence $\{Y, Y_n, n \geq 1\}$. Hence

$$\frac{\sum_{k=1}^{n} a_k Y_k}{B_n^*} \xrightarrow{d} N(0,1).$$

Then remembering that $B_n^* \sim B_n$,

$$\frac{\sum_{k=1}^{n} a_k(X_k-EX)}{B_n} = \left(\frac{B_n^*}{B_n}\right)\left(\frac{\sum_{k=1}^{n} a_k Y_k}{B_n^*}\right) \xrightarrow{d} N(0,1). \quad \square$$

2.5 Asymptotic Negligibility

In order to establish a CLT, "it is [generally] essential to impose a hypothesis where individual terms in the sum $S_n = \sum_{k=1}^{n} a_k X_k$ are 'negligible' in comparison with the sum itself" (Chung, 1974, p. 197). There are several different measures of negligibility. The literature (Loève, 1977, p. 302; Laha and Rohatgi, 1979, p. 295) calls the double array $\{a_k X_k/B_n,\ 1 \leq k \leq n,\ n \geq 1\}$ uniformly asymptotically negligible (u.a.n.) if

$$\lim_{n\to\infty} \max_{1\leq k\leq n} P\{|\frac{a_k X_k}{B_n}| > \varepsilon\} = 0 \quad \text{for all } \varepsilon > 0.$$

This condition is also known by various other names, such as infinitesimal (Chow and Teicher, 1978, p. 422) or holospoudic (Chung, 1974, p. 198).

Another measure of negligibility is

$$\lim_{n\to\infty} \max_{1\leq k\leq n} |\text{median}(\frac{a_k X_k}{B_n})| = 0. \tag{57}$$

Fact. If $\{a_k X_k/B_n,\ 1 \leq k \leq n,\ n \geq 1\}$ is u.a.n., then (57) holds.

Proof. Let $\varepsilon > 0$. By hypothesis, there exists an integer N such that if $n \geq N$

$$P\{|\frac{a_k X_k}{B_n}| > \varepsilon\} < \frac{1}{2} \quad \text{for all } k = 1, \ldots, n,$$

whence (57) is established. □

The measure of negligibility that we are using in our CLTs is the condition (10). Under our hypotheses, it is equivalent to

$$\max_{1\le k\le n} \frac{|a_k X_k|}{B_n} \xrightarrow{P} 0. \tag{58}$$

Fact. If $\{X_n, n \ge 1\}$ are i.i.d. random variables, then (10) and (58) are equivalent.

Proof. Let $\varepsilon > 0$ and define $p_{nk} = P\{|a_k X_k| > \varepsilon B_n\}$, $1 \le k \le n$, $n \ge 1$. Thus,

$$\begin{aligned} &P\{\max_{1\le k\le n} \frac{|a_k X_k|}{B_n} > \varepsilon\} \\ &= P\{\bigcup_{k=1}^{n} [|a_k X_k| > \varepsilon B_n]\} \\ &= 1-P\{\bigcap_{k=1}^{n} [|a_k X_k| \le \varepsilon B_n]\} \\ &= 1-\prod_{k=1}^{n} P\{|a_k X_k| \le \varepsilon B_n\} \\ &= 1-\prod_{k=1}^{n} [1-P\{|a_k X_k| > \varepsilon B_n\}] \\ &= 1-\prod_{k=1}^{n} (1-p_{nk}). \end{aligned}$$

Hence (58) holds if $1-\prod_{k=1}^{n}(1-p_{nk}) = o(1)$. Recalling the elementary inequality $1-x \le e^{-x}$,

$$0 \leq 1-\exp\{-\sum_{k=1}^{n} p_{nk}\} = 1-\prod_{k=1}^{n} \exp\{-p_{nk}\}$$

$$\leq 1-\prod_{k=1}^{n}(1-p_{nk}) \leq \sum_{k=1}^{n} p_{nk}.$$

Thus, $1-\prod_{k=1}^{n}(1-p_{nk}) = o(1)$ iff $\sum_{k=1}^{n} p_{nk} = o(1)$. Therefore (58) is equivalent to (10). □

Our measure of smallness (10) implies the u.a.n. condition.

Fact. If (10) holds, then $\{a_k X_k/B_n,\ 1 \leq k \leq n,\ n \geq 1\}$ is u.a.n.

Proof. Let $\varepsilon > 0$. Clearly, for all $k = 1, \ldots, n$

$$P\{|a_k X_k| > \varepsilon B_n\} \leq P\{\max_{1\leq j\leq n} |a_j X_j| > \varepsilon B_n\}.$$

Thus

$$\max_{1\leq k\leq n} P\{|a_k X_k| > \varepsilon B_n\}$$

$$\leq P\{\max_{1\leq j\leq n} |a_j X_j| > \varepsilon B_n\} = o(1) \qquad \text{(by (10))}. \ \square$$

2.6 An Asymptotic Representation for $\{B_n,\ n > 1\}$

The CLTs in this chapter hinge upon hypotheses that involve functions of regular variation. At first we assume that (8) holds; later we suppose that $P\{|X| > t\}$ is a regularly varying function. In either case G and H are slowly varying. We have seen via Lemma 2.3 that the random variable X is in $\mathcal{L}_p$ for all $0 < p < 2$, but $EX^2 = \infty$ is the case of greatest interest to us. It is natural to ask which random variables have a distribution which satisfies these hypotheses. To understand this class of random variables one needs to study the behavior of regularly varying functions. We know, by the Karamata Representation Theorem (46), that all slowly varying functions are of a specific form. Also, recall that a function W(t) is regularly varying with exponent ρ $(-\infty < \rho < \infty)$ if and only if $W(t) = t^{\rho}L(t)$ where L(t) is a slowly varying function. With all this in mind we will now generate an interesting class of slowly varying functions which are applicable to our CLTs with $EX^2 = \infty$.

Suppose the distribution of the random variable X satisfies for some $0 < \alpha < 1$

$$P\{|X| > t\} \sim \frac{\alpha \exp\{(\log t)^{\alpha}\}}{2t^2(\log t)^{1-\alpha}} \quad \text{as } t \to \infty.$$

Thus

$$G(x) \sim \int_e^x \frac{\alpha \exp\{(\log t)^{\alpha}\}}{t(\log t)^{1-\alpha}} dt \sim \exp\{(\log x)^{\alpha}\}.$$

One can check that this is a slowly varying function by observing that, for $s > 0$, $G(sx) \sim G(x)$ as $x \to \infty$. Alternatively, one can easily verify that (46) holds. For let $a(t) \equiv 1$ and $\varepsilon(t) = \alpha(\log t)^{\alpha-1}$. Then

$$\exp\{\int_1^t \frac{\alpha(\log s)^{\alpha-1}}{s} ds\} = \exp\{\int_0^{\log t} \alpha u^{\alpha-1} du\}$$

$$= \exp\{(\log t)^{\alpha}\} \sim G(t).$$

We require that $\alpha < 1$ so that $\varepsilon(t) \to 0$ and we also need $\alpha > 0$ to ensure that $G(t) \to \infty$.

From (13) we obtain $q_n = \sqrt{n}\sqrt{G(q_n)}$. Since G is slowly varying, a natural question is whether we can replace $G(q_n)$ by $G(\sqrt{n})$ and thus conclude that

$$q_n \sim \sqrt{n}\sqrt{G(\sqrt{n})} \tag{59}$$

thereby yielding, via (7), an explicit asymptotic representation for $\{B_n, n \geq 1\}$.

Define $G_\alpha(x) = \exp\{(\log x)^\alpha\}$, $0 < \alpha < 1$. This class of slowly varying functions has some intriguing properties. It will be seen that (59) holds for some, but not all, α in $(0,1)$.

Proposition 2.1. If $G_\alpha(x) = \exp\{(\log x)^\alpha\}$, then

$$\lim_{n\to\infty} \frac{q_n}{\sqrt{n}\sqrt{G_\alpha(\sqrt{n})}} = \begin{cases} 1 & \text{for } 0 < \alpha < 1/2, \\ e^{1/8} & \text{for } \alpha = 1/2, \\ \infty & \text{for } 1/2 < \alpha < 1. \end{cases}$$

Proof. We need to observe the limiting behavior of $(\log q_n)^\alpha - (\log \sqrt{n})^\alpha$. From (13) we see that $q_n = \sqrt{n}\sqrt{G_\alpha(q_n)}$, whence

$$\log q_n = \log \sqrt{n} + \tfrac{1}{2}\log G_\alpha(q_n)$$

$$= \log \sqrt{n} + \tfrac{1}{2}(\log q_n)^\alpha. \qquad (60)$$

Let $m = \log \sqrt{n}$ and note that for all large n, $q_n \geq \sqrt{n}$. Hence from (60), if n is sufficiently large,

$$(\log q_n)^\alpha - (\log \sqrt{n})^\alpha = (\log \sqrt{n} + \tfrac{1}{2}(\log q_n)^\alpha)^\alpha - (\log \sqrt{n})^\alpha$$

$$\geq (\log \sqrt{n} + \tfrac{1}{2}(\log \sqrt{n})^\alpha)^\alpha - (\log \sqrt{n})^\alpha$$

$$= (m + \tfrac{1}{2}m^\alpha)^\alpha - m^\alpha$$

$$= \alpha x_m^{\alpha-1}(\tfrac{1}{2}m^\alpha)$$

by the mean value theorem for some x_m where $m \leq x_m \leq m + \frac{1}{2}m^\alpha$. Therefore

$$(\log q_n)^\alpha - (\log \sqrt{n})^\alpha \geq \tfrac{\alpha}{2}(m + \tfrac{1}{2}m^\alpha)^{\alpha-1}m^\alpha$$

$$= (1+o(1))\tfrac{\alpha}{2}m^{\alpha-1}m^\alpha = (1+o(1))\tfrac{\alpha}{2}m^{2\alpha-1}.$$

Hence

$$\liminf_{n\to\infty} \{(\log q_n)^\alpha - (\log \sqrt{n})^\alpha\} \geq \tfrac{\alpha}{2} \lim_{m\to\infty} m^{2\alpha-1}. \qquad (61)$$

Since G_α is slowly varying we obtain by (3) for all $0 < \varepsilon < 1$ that $G(x) \leq x^{2\varepsilon}$ for all large x. Then $q_n^2 = nG(q_n) \leq nq_n^{2\varepsilon}$, which shows that $q_n \leq n^{\frac{1}{2}(\frac{1}{1-\varepsilon})}$ for all $0 < \varepsilon < 1$ and large n. So for arbitrary $\beta > 1$ and all large n, $q_n \leq n^{\beta/2}$. Therefore, recalling (60), if n is sufficiently large

$$(\log q_n)^\alpha - (\log \sqrt{n})^\alpha = (\log \sqrt{n} + \tfrac{1}{2}(\log q_n)^\alpha)^\alpha - (\log \sqrt{n})^\alpha$$

$$\leq (\log \sqrt{n} + \tfrac{1}{2}(\log n^{\beta/2})^\alpha)^\alpha - (\log \sqrt{n})^\alpha$$

$$= (\log \sqrt{n} + \frac{\beta^2}{2}(\log \sqrt{n})^\alpha)^\alpha - (\log \sqrt{n})^\alpha$$

$$= (m + \frac{\beta^\alpha}{2}m^\alpha)^\alpha - m^\alpha$$

$$= \alpha x_m^{\alpha-1}(\frac{\beta^\alpha}{2})m^\alpha$$

by the mean value theorem for some x_m where $m \leq x_m \leq m + \frac{\beta^\alpha}{2}m^\alpha$. Thus,

$$(\log q_n)^\alpha - (\log \sqrt{n})^\alpha \leq \alpha m^{\alpha-1}(\frac{\beta^\alpha}{2})m^\alpha = \frac{\alpha}{2}\beta^\alpha m^{2\alpha-1}.$$

This, in turn, yields for all $\beta > 1$ that

$$\limsup_{n\to\infty}\{(\log q_n)^{\alpha} - (\log \sqrt{n})^{\alpha}\} \leq \frac{\alpha}{2}\beta^{\alpha}\lim_{m\to\infty} m^{2\alpha-1}.$$

Letting $\beta \downarrow 1$ we obtain

$$\limsup_{n\to\infty}\{(\log q_n)^{\alpha} - (\log \sqrt{n})^{\alpha}\} \leq \frac{\alpha}{2}\lim_{m\to\infty} m^{2\alpha-1}.$$

This together with (61) yields

$$\lim_{n\to\infty}\{(\log q_n)^{\alpha} - (\log \sqrt{n})^{\alpha}\} = \frac{\alpha}{2}\lim_{m\to\infty} m^{2\alpha-1}$$

$$= \begin{cases} 0 & \text{for } 0 < \alpha < 1/2, \\ 1/4 & \text{for } \alpha = 1/2, \\ \infty & \text{for } 1/2 < \alpha < 1. \end{cases}$$

Therefore

$$\lim_{n\to\infty}\frac{q_n}{\sqrt{n}\sqrt{G_{\alpha}(\sqrt{n})}} = \lim_{n\to\infty}\frac{\sqrt{n}\sqrt{G_{\alpha}(q_n)}}{\sqrt{n}\sqrt{G_{\alpha}(\sqrt{n})}} = \lim_{n\to\infty}\left(\frac{G_{\alpha}(q_n)}{G_{\alpha}(\sqrt{n})}\right)^{1/2}$$

$$= \lim_{n\to\infty}(\exp\{(\log q_n)^{\alpha} - (\log \sqrt{n})^{\alpha}\})^{1/2}$$

$$= \begin{cases} 1 & \text{for } 0 < \alpha < 1/2, \\ e^{1/8} & \text{for } \alpha = 1/2, \\ \infty & \text{for } 1/2 < \alpha < 1. \quad \square \end{cases}$$

Remark. In this last example we let $G_\alpha(x) = \exp\{(\log x)^\alpha\}$ but if we let $\tilde{G}_\alpha(x) \sim G_\alpha(x)$, the conclusion of Proposition 2.1 will now be shown to be valid for $\tilde{G}_\alpha$.

Proof. Let $G_\alpha(x) = \exp\{(\log x)^\alpha\}$, $0 < \alpha < 1$, and let $\tilde{G}_\alpha(x)$ be some function defined by (5) with $\tilde{G}_\alpha(x) \sim G_\alpha(x)$. Define $\tilde{Q}_\alpha(x) = x^2/\tilde{G}_\alpha(x)$, $Q_\alpha(x) = x^2/G_\alpha(x)$, $\tilde{q}_\alpha(x) = \tilde{Q}_\alpha^{-1}(x)$, and $q_\alpha(x) = Q_\alpha^{-1}(x)$. Since $Q_\alpha(x) \sim \tilde{Q}_\alpha(x)$we have $Q_\alpha(q_\alpha(t)) = t = \tilde{Q}_\alpha(\tilde{q}_\alpha(t)) \sim Q_\alpha(\tilde{q}_\alpha(t))$, whence

$$Q_\alpha(q_\alpha(t)) \sim Q_\alpha(\tilde{q}_\alpha(t)). \tag{62}$$

Utilizing Lemma 2.7 with $Q_\alpha(x) = x^2L(x)$ (where $L(x) = 1/G_\alpha(x)$), we see that $y^{-1/2}q_\alpha(y)$ is slowly varying. Applying this to (62) we obtain, via Lemma 2.9, that

$$[Q_\alpha(q_\alpha(t))]^{-1/2} \cdot q_\alpha(Q_\alpha(q_\alpha(t))) \sim [Q_\alpha(\tilde{q}_\alpha(t))]^{-1/2} \cdot q_\alpha(Q_\alpha(\tilde{q}_\alpha(t))).$$

Reapplying (62) we can conclude that $q_\alpha(t) \sim \tilde{q}_\alpha(t)$. This, together with $\tilde{G}_\alpha(x) \sim G_\alpha(x)$, shows that

$$\lim_{n\to\infty} \frac{\tilde{q}_n}{\sqrt{n}\sqrt{\tilde{G}_\alpha(\sqrt{n})}} \qquad \text{(where } \tilde{q}_n \equiv \tilde{q}_\alpha(n))$$

$$= \lim_{n\to\infty} \frac{q_n}{\sqrt{n}\sqrt{G_\alpha(\sqrt{n})}} = \begin{cases} 1 & \text{for } 0 < \alpha < 1/2, \\ e^{1/8} & \text{for } \alpha = 1/2, \\ \infty & \text{for } 1/2 < \alpha < 1. \end{cases} \square$$

In view of (13) and the example in Proposition 2.1, an interesting question to raise is whether (59) prevails for a large class of distributions. The answer is affirmative in view of the ensuing Propositions 2.2 and 2.3.

Proposition 2.2. If $G(x) \to \infty$, then $G(q_n) \sim G(\sqrt{n})$ iff $G(x) \sim G(x\sqrt{G(x)})$. These, in turn, are equivalent to (59). In such a case, G is necessarily slowly varying.

Proof. Recall that $q_n = \sqrt{n}\sqrt{G(q_n)}$, whence $\sqrt{n} \leq q_n$ for n sufficiently large. Assuming that $G(q_n) \sim G(\sqrt{n})$ and utilizing the fact that G is nondecreasing we obtain for n sufficiently large that

$$G(\sqrt{n}) \leq G(\sqrt{n}\sqrt{G(\sqrt{n})}) \leq G(\sqrt{n}\sqrt{G(q_n)})$$

$$= G(q_n) = (1+o(1))G(\sqrt{n}).$$

Thus $G(\sqrt{n}) \sim (\sqrt{n}\sqrt{G(\sqrt{n})})$. Now to show $G(x) \sim G(x\sqrt{G(x)})$, apply Lemma 2.9 twice. First we obtain $G(\sqrt{n}) \sim G(\sqrt{n+1})$ and then $G(\sqrt{n}\sqrt{G(\sqrt{n})}) \sim G(\sqrt{n+1}\sqrt{G(\sqrt{n+1})})$. Hence, if $\sqrt{n} \leq x \leq \sqrt{n+1}$ we have

$$G(\sqrt{n}) \leq G(x) \leq G(x\sqrt{G(x)}) \leq G(\sqrt{n+1}\sqrt{G(\sqrt{n+1})})$$

$$= (1+o(1))G(\sqrt{n}\sqrt{G(\sqrt{n})}) = (1+o(1))G(\sqrt{n})$$

thereby proving the necessity portion.

Next, we prove the sufficiency half of this proposition. Again we use Lemma 2.9. By assumption $G(x) \sim G(x\sqrt{G(x)})$ implying that

$$x \sim \frac{x\sqrt{G(x)}}{\sqrt{G(x\sqrt{G(x)})}} \to \infty.$$

Thus, by Lemma 2.9

$$G(x) \sim G\left(\frac{x\sqrt{G(x)}}{\sqrt{G(x\sqrt{G(x)})}}\right).$$

Therefore

$$G(x\sqrt{G(x)}) \sim G\left(\frac{x\sqrt{G(x)}}{\sqrt{G(x\sqrt{G(x)})}}\right)$$

whence replacing $x\sqrt{G(x)}$ with q_n yields

$$G(q_n) \sim G\left(\frac{q_n}{\sqrt{G(q_n)}}\right) = G(\sqrt{n}) \qquad \text{(by (13))}.$$

Finally, we will show that G is slowly varying. If $s \geq 1$, then for sufficiently large x

$$G(x) \leq G(sx) \leq G(x\sqrt{G(x)})$$

which, together with $G(x) \sim G(x\sqrt{G(x)})$, shows that $G(x) \sim G(sx)$. If $0 < s < 1$, then by the part already proved

$$G(x) = G\left(\frac{1}{s}(sx)\right) \sim G(sx).$$

Thus $G(sx) \sim G(x)$ as $x \to \infty$ for all $s > 0$ and so G is slowly varying. $\square$

With Proposition 2.2 in mind we would like to find other conditions that imply (59) or, equivalently, $G(x) \sim G(x\sqrt{G(x)})$. Before we do so, we need to state a definition and prove a preliminary lemma. We say that a nonnegative function g, defined on $[0,\infty)$, preserves asymptotic equivalence at infinity if $g(x_n) \sim g(y_n)$ whenever $\{x_n, n \geq 1\}$ and $\{y_n, n \geq 1\}$ are nonnegative sequences with $x_n \sim y_n \to \infty$. We have already noted, in Lemma 2.9, that if L is slowly varying and $a(t) \sim b(t) \to \infty$, then $L(a(t)) \sim L(b(t))$. We now establish the following lemma.

Lemma 2.14. Let g be a nonnegative, nondecreasing function defined on $[0,\infty)$ which preserves asymptotic equivalence at infinity. Let $\bar{g}$ defined on $[0,\infty)$ agree with g on the integers and be defined by linear interpolation between the integers, i.e.,

$$\bar{g}(x) = (g(n+1)-g(n))(x-n) + g(n) \text{ for } n \leq x \leq n+1,\ n \geq 0. \qquad (63)$$

Then $\bar{g}(x) \sim g(x)$ as $x \to \infty$ and $\bar{g}$ preserves asymptotic equivalence at infinity.

Proof. For all large x, writing $n \leq x \leq n+1$, it follows from the hypotheses on g that

$$1+o(1) \leq \frac{g(n)}{g(n+1)} = \frac{\bar{g}(n)}{g(n+1)} \leq \frac{\bar{g}(x)}{g(x)} \leq \frac{\bar{g}(n+1)}{g(n)} = \frac{g(n+1)}{g(n)}$$

$$= 1+o(1) \qquad\qquad \text{as } x \to \infty,$$

whence $\bar{g}(x) \sim g(x)$ as $x \to \infty$. Then if $x_n \sim y_n \to \infty$

$$\bar{g}(x_n) \sim g(x_n) \sim g(y_n) \sim \bar{g}(y_n)$$

and so $\bar{g}$ preserves asymptotic equivalence at infinity.$\square$

The next proposition, when combined with Proposition 2.2, (7), and (13) yields the explicit asymptotic representation

$B_n \sim s_n\sqrt{G(\sqrt{n})}$.

Proposition 2.3. Let G be defined as in (5) and suppose that $G(x) \to \infty$. Then (i) => (ii) => (iii) => $G(x) \sim G(x\sqrt{G(x)})$ (or, equivalently, $B_n \sim s_n\sqrt{G(\sqrt{n})}$) where (i), (ii), (iii) are given by:

(i) there exists $0 < t(x) \uparrow$, $0 < b(x) \uparrow$ such that

$$G(x) \sim \frac{\exp\{\frac{\log_2 x}{t(x)}\}}{b(x)};$$

(ii) $G(u_n) \sim G(v_n)$ whenever $\{u_n, n \geq 1\}$ and $\{v_n, n \geq 1\}$ are real sequences with $\log u_n \sim \log v_n \to \infty$;

(iii) there exists $r(x) \uparrow \infty$, $s(x) \uparrow$ such that

$$G(x) \sim \frac{\exp\{\frac{\sqrt{\log x}}{r(x)}\}}{s(x)}.$$

Moreover, $G(x) \sim G(x\sqrt{G(x)})$ implies that G is slowly varying.

Proof. Suppose that (i) holds and let $\log u_n \sim \log v_n \to \infty$. Set $m_n = \min\{u_n, v_n\}$, $M_n = \max\{u_n, v_n\}$, $n \geq 1$. Then

$$1 \leq \frac{G(M_n)}{G(m_n)} = (1+o(1))\frac{b(m_n)}{b(M_n)}\exp\{\frac{\log_2 M_n}{t(M_n)} - \frac{\log_2 m_n}{t(m_n)}\}$$

$$\leq (1+o(1))\exp\{\frac{\log_2 M_n - \log_2 m_n}{t(m_n)}\}$$

$$= (1+o(1))\exp\{O(1)\log(\frac{\log M_n}{\log m_n})\}$$

$$= (1+o(1))\exp\{O(1)|\log(\frac{\log u_n}{\log v_n})|\}$$

$$= 1+o(1)$$

implying that $G(m_n) \sim G(M_n)$. Therefore,

$$1+o(1) = \frac{G(m_n)}{G(M_n)} \leq \frac{G(u_n)}{G(v_n)} \leq \frac{G(M_n)}{G(m_n)} = 1+o(1)$$

thereby proving (i) => (ii).

Next suppose that (ii) holds. It will be shown that (iii) obtains with $r(x) = (\log x)^{1/4}$. Define $g(x) = G(\exp\{x^4\})$, $x \geq 0$. We will now verify that g preserves asymptotic equivalence at infinity. Let $x_n \sim y_n \to \infty$ and set $u_n = \exp\{x_n^4\}$, $v_n = \exp\{y_n^4\}$, $n \geq 1$. Then $\log u_n = x_n^4 \sim y_n^4 = \log v_n \to \infty$ implying via (ii) that

$$g(x_n) = g((\log u_n)^{1/4}) = G(u_n) \sim G(v_n)$$

$$= g((\log v_n)^{1/4}) = g(y_n).$$

Define $\bar{g}$ on $[0,\infty)$ as in (63). By Lemma 2.14, $\bar{g}(x) \sim g(x)$ and $\bar{g}$ preserves asymptotic equivalence at infinity. Thus,

$$\frac{\exp\{(\log x)^{1/4}\}}{G(x)} = \frac{\exp\{(\log x)^{1/4}\}}{g((\log x)^{1/4})}$$

$$\sim \frac{\exp\{(\log x)^{1/4}\}}{\bar{g}((\log x)^{1/4})}$$

and to complete the proof it suffices to show that $e^x/\bar{g}(x)$ is eventually nondecreasing. To this end, observe that it suffices to show for n sufficiently large and $0 \leq t_1 \leq t_2 \leq 1$ that

$$\frac{e^{n+t_2}}{\bar{g}(n+t_2)} \geq \frac{e^{n+t_1}}{\bar{g}(n+t_1)}. \tag{64}$$

Choose N such that $\bar{g}(n+1)/\bar{g}(n) \leq 2$ for all $n \geq N$. For $n \geq N$ and $0 < t < 1$, set

$$\Lambda_n(t) = \frac{e^{n+t}}{\bar{g}(n+t)}.$$

For $0 \leq t_1 \leq t_2 \leq 1$, Λ_n is differentiable on (t_1,t_2) and is right and left continuous at t_1, t_2, respectively. Now on (t_1,t_2),

$$\Lambda_n'(t) = \frac{\bar{g}(n+t)\, e^{n+t} - e^{n+t}\, \bar{g}'(n+t)}{[\bar{g}(n+t)]^2}$$

$$= \frac{e^{n+t}}{[\bar{g}(n+t)]^2}[\bar{g}(n+t) - (g(n+1) - g(n))]$$

$$= \frac{e^{n+t}}{[\bar{g}(n+t)]^2}[\bar{g}(n+t) - \bar{g}(n+1) + \bar{g}(n)]$$

$$\geq \frac{e^{n+t}}{[\bar{g}(n+t)]^2}[2\bar{g}(n) - \bar{g}(n+1)] \geq 0$$

recalling the choice of N. The mean value theorem guarantees the existence of a point Θ_n in (t_1,t_2) such that

$$\Lambda_n(t_2) - \Lambda(t_1) = \Lambda_n'(\Theta_n)(t_2-t_1).$$

However $t_2 > t$, and $\Lambda_n'(t) \geq 0$, whence $\Lambda_n(t_2) \geq \Lambda_n(t_1)$ thereby proving (64).

Now, suppose that (iii) holds. Then

$$1 \leq \frac{G(x\sqrt{G(x)})}{G(x)} = (1+o(1))\frac{s(x)}{s(x\sqrt{G(x)})}\exp\{\frac{\sqrt{\log(x\sqrt{G(x)})}}{r(x\sqrt{G(x)})} - \frac{\sqrt{\log x}}{r(x)}\}$$

$$\leq (1+o(1))\exp\{\frac{\sqrt{\log(x\sqrt{G(x)})} - \sqrt{\log x}}{r(x)}\}. \qquad (65)$$

But

$$\log(x\sqrt{G(x)}) = \log x + \frac{1}{2}\log G(x)$$

$$= \log x + \frac{1}{2}(1+o(1))\left(\frac{\sqrt{\log x}}{r(x)} - \log s(x)\right) \qquad \text{(by Lemma 2.9)}$$

$$= [\log x]\left[1+ \frac{1}{2}(1+o(1))\left(\frac{1}{r(x)\sqrt{\log x}} - \frac{\log s(x)}{\log x}\right)\right]$$

implying for all large x that

$$\sqrt{\log(x\sqrt{G(x)})} = \sqrt{\log x}\left[1+ \frac{1}{2}(1+o(1))\left(\frac{1}{r(x)\sqrt{\log x}} - \frac{\log s(x)}{\log x}\right)\right]^{1/2}$$

$$\leq \sqrt{\log x}\left(1+ \frac{1}{r(x)\sqrt{\log x}} - \frac{\log s(x)}{\log x}\right)^{1/2} \qquad \text{(since } G(x) \to \infty)$$

$$\leq \sqrt{\log x}\left(1+ \frac{1}{r(x)\sqrt{\log x}} + \frac{1}{\sqrt{\log x}}\right)^{1/2} \qquad \text{(since } 0 < s(x) \uparrow)$$

$$\leq \sqrt{\log x}\left(1+ \frac{1}{r(x)\sqrt{\log x}} + \frac{1}{\sqrt{\log x}}\right)$$

$$= \sqrt{\log x} + \frac{1}{r(x)} + 1 \leq \sqrt{\log x} + 2 \qquad \text{(since } r(x) \uparrow \infty).$$

Hence for all large x,

$$\sqrt{\log(x\sqrt{G(x)})} - \sqrt{\log x} \leq 2.$$

Then, from (65), for all large x

$$1 \leq \frac{G(x\sqrt{G(x)})}{G(x)} \leq (1+o(1))\exp\{\frac{2}{r(x)}\}$$

$$= 1+o(1)) \qquad \text{(since } r(x) \uparrow \infty\text{)}.$$

thereby proving $G(x) \sim G(x\sqrt{G(x)})$.

Finally, the last assertion was proved in Proposition 2.2. □

2.7 Examples

We conclude this chapter with two examples illustrating some of the results of the chapter.

Example 2.1. Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables with common density function

$$f(x) = \begin{cases} 2x^{-3} & \text{for } x > 1, \\ 0 & \text{for } x \leq 1. \end{cases}$$

Then

$$\frac{\sum_{k=1}^{n} \sqrt{\log k}(X_k - 2)}{\sqrt{n}\log n} \xrightarrow{d} N(0,1).$$

Proof. Note that $EX = 2$ and $EX^2 = \infty$. Also, for $x \geq 1$,

$$P\{|X| > x\} = \int_x^\infty 2t^{-3}dt = x^{-2}. \qquad (66)$$

Thus, $P\{|X| > x\}$ is regularly varying with exponent -2. Then, for $x \geq 1$,

$$G(x) = \int_0^x 2tP\{|X| > t\}dt$$

$$= \int_0^1 2tdt + \int_1^x 2t^{-1}dt$$

$$= 1 + 2\log x$$

which is a slowly varying function.

Set $a_n = \sqrt{\log n}$, $n \geq 1$. Then

$$s_n^2 = \sum_{k=1}^n \log k \sim \int_1^n (\log x)dx \sim \int_1^n (1+\log x)dx$$

$$= n\log n.$$

Note, that condition (ii) of Proposition 2.3 is satisfied, whence

$$B_n \sim s_n\sqrt{G(\sqrt{n})} \sim \sqrt{n}\log n.$$

Next observe that $\min\limits_{1\leq k\leq n} |a_k| = 1$ and $\max\limits_{1\leq k\leq n} |a_k| = \sqrt{\log n}$. Set, for $n \geq 1$,

$$\alpha_n = \frac{B_n}{\min\limits_{1\leq k\leq n} |a_k|} \sim \sqrt{n}\log n$$

and

$$\beta_n = \frac{B_n}{\max\limits_{1\le k\le n} |a_k|} \sim \sqrt{n\log n}.$$

Thus, recalling Lemma 2.9,

$$G(\beta_n) \le G(\alpha_n) = (1+o(1))G(\sqrt{n}\log n)$$

$$= (1+o(1))\log n = (1+o(1))G(\sqrt{n\log n}) = (1+o(1))G(\beta_n)$$

and so $G(\beta_n) \sim G(\alpha_n)$ establishing (55).

Finally, let $\varepsilon > 0$. For n sufficiently large

$$\sum_{k=1}^{n} P\{|X| > \varepsilon B_n/|a_k|\}$$

$$\le \sum_{k=1}^{n} P\{|X| > \frac{\varepsilon\sqrt{n}\log n}{2\sqrt{\log k}}\} \qquad \text{(since } B_n \sim \sqrt{n}\log n)$$

$$= \sum_{k=1}^{n} \frac{4\log k}{\varepsilon^2 n(\log n)^2} \qquad \text{(by (66))}$$

$$\le \frac{4n\log n}{\varepsilon^2 n(\log n)^2} = \frac{4}{\varepsilon^2 \log n} = o(1)$$

establishing (56). Then by Theorem 2.5,

$$\frac{\sum\limits_{k=1}^{n} \sqrt{\log k}(X_k-2)}{B_n} \xrightarrow{d} N(0,1).$$

Since $B_n \sim \sqrt{n}\log n$, the desired result follows. □

This next example illustrates Corollary 2.1. It will also create a family of CLTs, since the weights $\{a_n, n \geq 1\}$ are not explicitly defined.

Example 2.2. Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables with common density function

$$f(x) = \begin{cases} \frac{e^2}{2}\left(\frac{2(\log|x|)-1}{|x|^3}\right) & \text{for } |x| \geq e, \\ 0 & \text{for } |x| < e. \end{cases}$$

Let L be any positive slowly varying and nondecreasing function. Then

$$\frac{\sum_{k=1}^{n} L(k)X_k}{e\sqrt{n}L(n)\log\sqrt{n}} \xrightarrow{d} N(0,1).$$

Proof. Firstly, via integration by parts, we verify that f is indeed a density. For

$$\int_{-\infty}^{\infty} f(x)dx = e^2\int_e^{\infty} \frac{2(\log x)-1}{x^3}dx$$

$$= e^2[(2(\log x)-1)(-\frac{1}{2}x^{-2})\Big|_e^{\infty} + \int_e^{\infty} x^{-3}dx]$$

$$= e^2[\frac{1}{2}e^{-2} + \frac{1}{2}e^{-2}] = 1.$$

Clearly $EX = 0$ and $EX^2 = \infty$. For $x \geq e$,

$$P\{|X| > x\} = e^2\int_x^\infty \left(\frac{2(\log t)-1}{t^3}\right)dt$$

$$= e^2\left[(2(\log t)-1)\left(-\frac{1}{2}t^{-2}\right)\Big|_x^\infty + \int_x^\infty t^{-3}dt\right]$$

$$= \left(\frac{e}{x}\right)^2\log x$$

and so $P\{|X| > x\}$ is regularly varying with exponent -2. For $x \geq e$,

$$G(x) = \int_0^e 2t\,dt + \int_e^x 2t\left(\frac{e}{t}\right)^2(\log t)dt$$

$$= (e\log x)^2$$

which is slowly varying. Note that since L is slowly varying, it follows that (see, e.g., Feller, 1971, p. 281)

$$s_n^2 = \sum_{k=1}^n a_k^2 = \sum_{k=1}^n L^2(k) \sim nL^2(n).$$

Also,

$$G(x\sqrt{G(x)}) = G(xe\log x) = e^2(\log(xe\log x))^2$$

$$\sim e^2(\log x)^2 = G(x).$$

Then by Proposition 2.2 or 2.3

$$B_n \sim s_n\sqrt{G(\sqrt{n})} \sim e\sqrt{n}L(n)\log \sqrt{n}.$$

Let $\varepsilon > 0$ and note that if n is sufficiently large $B_n \geq \sqrt{n}L(n)\log \sqrt{n}$. Hence for all large n

$$\sum_{k=1}^{n} P\{|X| > \varepsilon B_n/|a_k|\} = \sum_{k=1}^{n} P\{|X| > \varepsilon B_n/L(k)\}$$

$$\leq \sum_{k=1}^{n} P\{|X| > \varepsilon B_n/L(n)\} \qquad \text{(since } L(x) \uparrow)$$

$$\leq nP\{|X| > \varepsilon\sqrt{n}\log \sqrt{n}\}$$

$$= \frac{ne^2}{\varepsilon^2 n(\log \sqrt{n})^2} \log(\varepsilon\sqrt{n}\log \sqrt{n}) = o(1)$$

whence (10) obtains. It remains to show that (9) holds.

Note that $\min_{1\leq k\leq n} |a_k| = L(1)$ and $\max_{1\leq k\leq n} |a_k| = L(n)$. Since G is slowly varying and $B_n \sim e\sqrt{n}L(n)\log \sqrt{n}$, Lemma 2.9 yields

$$G(B_n/\min_{1\leq k\leq n} |a_k|) \sim G(\frac{e\sqrt{n}L(n)\log \sqrt{n}}{L(1)})$$

$$\sim G(\sqrt{n}L(n)\log n)$$

and

$$G(B_n / \max_{1\le k\le n} |a_k|) \sim G(\frac{e\sqrt{n}L(n)\log\sqrt{n}}{L(n)})$$

$$\sim G(\sqrt{n}\log n).$$

Let $\alpha_n = \sqrt{n}L(n)\log n$, $n \ge 1$, and $\beta_n = \sqrt{n}\log n$, $n \ge 1$. To establish (9) we need to show that $G(\alpha_n) \sim G(\beta_n)$. Note that

$$G(\alpha_n) = e^2(\log(\sqrt{n}L(n)\log n))^2$$

$$= e^2(\frac{1}{2}(\log n)+(\log L(n))+\log_2 n)^2$$

$$\sim e^2(\frac{1}{2}\log n)^2 \qquad \text{(by (3))}$$

and

$$G(\beta_n) = e^2(\log(\sqrt{n}\log n))^2$$

$$= e^2(\frac{1}{2}(\log n)+\log_2 n)^2$$

$$\sim e^2(\frac{1}{2}\log n)^2$$

and so (9) obtains. Then, via Corollary 2.1,

$$\frac{\sum_{k=1}^{n} L(k)X_k}{e\sqrt{n}L(n)\log\sqrt{n}} = \frac{\sum_{k=1}^{n} L(k)X_k}{(1+o(1))B_n} \xrightarrow{d} N(0,1). \;\square$$

CHAPTER THREE
GENERALIZED STRONG LAWS OF LARGE NUMBERS

3.1 Introduction

In this chapter, we present generalized strong laws of large number (GSLLN) for weighted sums of random variables, i.e.,

$$\frac{\sum_{k=1}^{n} a_k(X_k-\gamma_k)}{b_n} \to 0 \quad \text{a.s.} \tag{1}$$

The hypotheses of these theorems vary greatly. In general, we control the behavior of the random variables $\{X_n, n \geq 1\}$ by restricting the magnitude of the tail of the distribution of $|X_n|$. Most of the assumptions that involve the sequences $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ only depend on the absolute value of their ratio. It thus proves convenient to define the sequence $\{c_n, n \geq 1\}$ by

$$c_n = b_n/|a_n|, \quad n \geq 1. \tag{2}$$

This notation will be used throughout this chapter.

The sequence $\{\gamma_n, n \geq 1\}$ will, for the most part, be the null sequence. In the remaining cases, $\{\gamma_n, n \geq 1\}$ will be a sequence of conditional or unconditional expectations.

3.2 Preliminary Lemmas

The theorems in this chapter assume that the random variables $\{X_n, n \geq 1\}$ are either i.i.d. or stochastically dominated. We say that the sequence $\{|X_n|, n \geq 1\}$ is stochastically dominated by $|X|$ if there exists a constant $D < \infty$ such that

$$P\{|X_n| > t\} \leq DP\{|X| > t/D\} \qquad t \geq 0,\ n \geq 1. \tag{3}$$

It is important to note that condition (3) does not place any restrictions on the joint distributions of the random variables $\{X_n\}$. Also it should be clear that if D satisfies (3), then any number larger than D also satisfies (3). Finally, note that if the random variables $\{X_n, n \geq 1\}$ are i.i.d., then (3) holds with $X = X_1$ and $D = 1$.

Lemma 3.1. Let X_0 and X be random variables such that $|X_0|$ is stochastically dominated by $|X|$ in the sense that there exists a constant $D_0 < \infty$ such that

$$P\{|X_0| > t\} \leq D_0 P\{|X| > t/D_0\} \qquad \text{for all } t \geq 0. \tag{4}$$

Then for all $q \geq 0$ and $s \geq 0$

$$E|X_0|^q I(|X_0| \leq s) \leq D_0^{q+1} E|X|^q I(|X| \leq s/D_0) + D_0 s^q P\{|X| > s/D_0\}.$$

Proof. Note that

$$E|X_0|^q I(|X_0| \le s) \le s^q P\{|X_0| > s\} + E|X_0|^q I(|X_0| \le s)$$

$$= q\int_0^s t^{q-1} P\{|X_0| > t\}dt \qquad \text{(by Lemma 2.1)}$$

$$\le qD_0\int_0^s t^{q-1} P\{|X| > t/D_0\}dt \qquad \text{(by (4))}$$

$$= qD_0\int_0^s t^{q-1} P\{|D_0X| > t\}dt$$

$$= D_0[s^q P\{|D_0X| > s\} + E|D_0X|^q I(|D_0X| \le s)] \quad \text{(by Lemma 2.1)}$$

$$= D_0 s^q P\{|X| > s/D_0\} + D_0^{q+1} E|X|^q I(|X| \le s/D_0). \ \square$$

Lemma 3.1 will be used in establishing Lemma 3.2 which, as will become apparent, plays a major role in this chapter.

Lemma 3.2. Let $\{X_n, n \ge 1\}$ and X be random variables such that $\{|X_n|, n \ge 1\}$ is stochastically dominated by $|X|$. Let $\{c_n, n \ge 1\}$ be constants with $c_n > 0$, where

$$\max_{1\le k\le n} c_k^q \sum_{j=n}^{\infty} \frac{1}{c_j^q} = O(n) \qquad \text{for some } q > 0 \tag{5}$$

and

$$\sum_{n=1}^{\infty} P\{|X| > Dc_n\} < \infty \tag{6}$$

where D is as in (3). Then for all $0 < M < \infty$

$$\sum_{k=1}^{\infty} \frac{1}{c_k^q} E|X_k|^q I(|X_k| \leq Mc_k) < \infty. \tag{7}$$

Proof. Let $0 < M < \infty$, let $D_o = \max\{D, \sqrt{M}\}$, $d_n = \max_{1 \leq k \leq n} c_k$ for $n \geq 1$, and set $d_0 = 0$. Note that $c_n \leq d_n$, $n \geq 1$, and (5) ensures that $d_n \uparrow \infty$.

In view of (5), there exists a constant $C < \infty$ such that

$$d_n^q \sum_{j=n}^{\infty} \frac{1}{c_j^q} \leq Cn \qquad \text{for all } n \geq 1. \tag{8}$$

Note that $D \leq D_o$ so (3) holds with D_o and then

$$\sum_{k=1}^{\infty} \frac{1}{c_k^q} E|X_k|^q I(|X_k| \leq Mc_k)$$

$$\leq \sum_{k=1}^{\infty} \frac{1}{c_k^q} E|X_k|^q I(|X_k| \leq D_o^2 c_k)$$

$$\leq \sum_{k=1}^{\infty} \frac{1}{c_k^q} [D_o^{q+1} E|X|^q I(|X| \leq D_o c_k) + D_o^{2q+1} c_k^q P\{|X| > D_o c_k\}]$$

(by Lemma 3.1 with $s = D_o^2 c_k$)

$$= D_o^{q+1} \sum_{k=1}^{\infty} \frac{1}{c_k^q} E|X|^q I(|X| \leq D_o c_k) + D_o^{2q+1} \sum_{k=1}^{\infty} P\{|X| > D_o c_k\}. \tag{9}$$

The series in the second term of (9) converges since it is bounded above by

$$D_o^{2q+1} \sum_{k=1}^{\infty} P\{|X| > Dc_k\} < \infty \qquad \text{(by (6))}.$$

The series in the first term of (9) is majorized by

$$\sum_{k=1}^{\infty} \frac{1}{c_k^q} E|X|^q I(|X| \le D_o d_k)$$

$$= \sum_{k=1}^{\infty} \frac{1}{c_k^q} \sum_{n=1}^{k} E|X|^q I(D_o d_{n-1} < |X| \le D_o d_n)$$

$$= \sum_{n=1}^{\infty} E|X|^q I(D_o d_{n-1} < |X| \le D_o d_n) \sum_{k=n}^{\infty} \frac{1}{c_k^q}$$

$$\le \sum_{n=1}^{\infty} P\{D_o d_{n-1} < |X| \le D_o d_n\} D_o^q d_n^q \sum_{k=n}^{\infty} \frac{1}{c_k^q}$$

$$\le CD_o^q \sum_{n=1}^{\infty} nP\{D_o d_{n-1} < |X| \le D_o d_n\} \qquad \text{(by (8))}$$

$$= C \sum_{n=1}^{\infty} \sum_{k=1}^{n} P\{D_o d_{n-1} < |X| \le D_o d_n\}$$

$$= C \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} P\{D_o d_{n-1} < |X| \le D_o d_n\}$$

$$= C \sum_{k=1}^{\infty} P\{|X| > D_0 d_{k-1}\}$$

$$\leq C[1 + \sum_{k=1}^{\infty} P\{|X| > D_0 d_k\}]$$

$$\leq C[1 + \sum_{k=1}^{\infty} P\{|X| > Dc_k\}] < \infty \qquad \text{(by (6))}$$

thereby proving the lemma. □

As previously noted, if the sequence $\{X_n, n \geq 1\}$ is identically distributed then (3) is automatic, but this is hardly necessary. It is quite easy to show that if the sequence $\{X_n, n \geq 1\}$ belongs to a scale family then (3) holds (subject to an additional assumption) as follows. (The symbol $\overset{d}{=}$ denotes that the two random variables have the same distribution.)

Remark. Let $X_n \overset{d}{=} \sigma_n X$, $n \geq 1$, $\sigma_n > 0$, and $\sup_{n\geq 1} \sigma_n < \infty$. Then $\{|X_n|, n \geq 1\}$ is stochastically dominated by $|X|$.

Proof. Let $D = \max\{1, \sup_{n\geq 1} \sigma_n\}$. Then for all $n \geq 1$, $t > 0$

$$P\{|X_n| > t\} = P\{|\sigma_n X| > t\}$$

$$= P\{|X| > t/\sigma_n\} \leq P\{|X| > \frac{t}{\sup_{n\geq 1} \sigma_n}\}$$

$$\leq DP\{|X| > t/D\}. \ \square$$

A well-known example of a scale family is the class of mean zero normal random variables.

Equation (6) compares the magnitude of the tail of the distribution of the random variables and the constants. It is well known (see, e.g., Chow and Teicher, 1978, pp. 89-90) that whenever $0 \le d_n \uparrow \infty$ (strictly), for every random variables X

$$\sum_{n=1}^{\infty} P\{|X| \ge d_n\} \le Ed^{-1}(|X|) \le \sum_{n=0}^{\infty} P\{|X| > d_n\} \tag{10}$$

where $d(\cdot)$ is a strictly monotone extension of $\{d_n, n \ge 1\}$ to $[0,\infty)$. Hence, in some situations the question as to whether or not (6) holds is immediate since, for example, from (10) we obtain

$$X \in \mathcal{L}_p \quad \text{iff} \quad \sum_{n=1}^{\infty} P\{|X| > n^{1/p}\} < \infty. \tag{11}$$

Finally, we need to comment on condition (5). It is clear that if $\{c_n, n \ge 1\}$ is of the form $c_n = n^{\alpha}$, for some $\alpha > 0$, then (5) obtains iff $q > \frac{1}{\alpha}$. The question at hand is when does (5) hold for an arbitrary $\{c_n, n \ge 1\}$? The next three lemmas address this question.

Lemma 3.3. If $0 < c_n \uparrow$ and $q > p > 0$, then

$$\sum_{k=n}^{\infty} \frac{1}{c_k^p} = O(n/c_n^p)$$

implies

$$\sum_{k=n}^{\infty} \frac{1}{c_k^q} = O(n/c_n^q).$$

Proof. Note that $c_n \uparrow$ and $q-p > 0$ ensure that $\frac{1}{c_k^{q-p}} \leq \frac{1}{c_n^{q-p}}$, $k \geq n \geq 1$. Therefore if

$$\sum_{k=n}^{\infty} \frac{1}{c_k^p} = O(n/c_n^p),$$

then

$$\sum_{k=n}^{\infty} \frac{1}{c_k^q} = \sum_{k=n}^{\infty} \frac{1}{c_k^p \cdot c_k^{q-p}} \leq \frac{1}{c_n^{q-p}} \sum_{k=n}^{\infty} \frac{1}{c_k^p}$$

$$= \frac{1}{c_n^{q-p}} O(n/c_n^p) = O(n/c_n^q). \ \square$$

Lemma 3.4. If $0 < |a_n| \downarrow$, $0 < b_n \uparrow$, and

$$b_n^q \sum_{j=n}^{\infty} \frac{1}{b_j^q} = O(n) \qquad \text{for some } q > 0,$$

then $b_n \uparrow \infty$ and (5) holds with the same q.

Proof. Since $b_n^q \sum_{j=n}^{\infty} \frac{1}{b_j^q} = O(n)$ we obtain $b_n \to \infty$, whence $b_n \uparrow \infty$.

Clearly $c_n = b_n/|a_n| \uparrow$ and

$$\max_{1 \leq k \leq n} c_k^q \sum_{j=n}^{\infty} \frac{1}{c_j^q} = c_n^q \sum_{j=n}^{\infty} \frac{1}{c_j^q} = \left(\frac{b_n}{|a_n|}\right)^q \sum_{j=n}^{\infty} \left(\frac{|a_j|}{b_j}\right)^q$$

$$\leq b_n^q \sum_{j=n}^{\infty} \frac{1}{b_j^q} = O(n)$$

thereby proving the lemma. □

Lemma 3.5. Suppose $0 < c_n^q/n \uparrow$ for some $q > 0$. Then (5) holds iff

$$\liminf_{n\to\infty} \frac{c_{rn}}{c_n} > r^{1/q} \qquad \text{for some integer } r \geq 2. \tag{12}$$

Proof. Let $d_n = c_n^q/n$, $n \geq 1$. Then $d_n \uparrow$ and $c_n = n^{1/q} d_n^{1/q}$, $n \geq 1$. So (12) is equivalent to

$$\liminf_{n\to\infty} \frac{(rn)^{1/q}(d_{rn})^{1/q}}{n^{1/q} d_n^{1/q}} > r^{1/q},$$

for some integer $r \geq 2$. This, in turn, is equivalent to

$$\liminf_{n\to\infty} \frac{d_{rn}}{d_n} > 1,$$

for some integer $r \geq 2$. On the other hand, (5) is equivalent to

$$\sum_{k=n}^{\infty} \frac{1}{kd_k} = O(\frac{1}{d_n}).$$

Hence we need only show that, whenever $0 < d_n \uparrow$,

$$\sum_{k=n}^{\infty} \frac{1}{kd_k} = O(\frac{1}{d_n})$$

is equivalent to

$$\liminf_{n\to\infty} \frac{d_{rn}}{d_n} > 1 \qquad \text{for some integer } r \geq 2.$$

This equivalence was proved by Martikainen (1985).$\square$

The following lemma is quite useful in our work. From (5) it is clear that $0 < c_n \to \infty$, but if $c_n/n^\alpha \uparrow$ for some $\alpha > 0$, then (6) can be weakened.

Lemma 3.6. Let X be any random variable and let $\{c_n, n \geq 1\}$ be constants such that $0 < c_n/n^\alpha \uparrow$ for some $\alpha > 0$. Then either

$$\sum_{n=1}^{\infty} P\{|X| > \lambda c_n\} < \infty \qquad \text{for all } \lambda > 0$$

or else

$$\sum_{n=1}^{\infty} P\{|X| > \lambda c_n\} = \infty \qquad \text{for all } \lambda > 0.$$

Proof. See Stout (1974, p. 131-132) or, for a new simple proof, see Rosalsky (1985). $\square$

Thus, if $0 < c_n/n^\alpha \uparrow$ for some $\alpha > 0$, then in order to verify (6) we need only check that

$$\sum_{n=1}^{\infty} P\{|X| > Ac_n\} < \infty \qquad \text{for some } A > 0.$$

3.3 Generalized Strong Laws of Large Numbers for Weighted Sums of Stochastically Dominated Random Variables

With these preliminaries accounted for, the first major result of this chapter may now be established. It is unfortunate that indicators are present in μ_n of the conclusion (15) of Theorem 3.1. However, under additional conditions, it is shown that (18) holds wherein the ν_n do not involve indicator functions.

Theorem 3.1. Let $\{X_n, n \geq 1\}$ and X be random variables such that $\{|X_n|, n \geq 1\}$ is stochastically dominated by $|X|$ in the sense that there exists constants $D_i < \infty$, $i = 1,2$ such that

$$P\{|X_n| > t\} \leq D_1 P\{|X| > t/D_2\} \qquad t \geq 0,\ n \geq 1. \tag{13}$$

Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be constants satisfying $0 < b_n \uparrow \infty$ and (5) with $q = 2$, where c_n is defined by (2). If

$$\sum_{n=1}^{\infty} P\{|X| > D_3 c_n\} < \infty \tag{14}$$

for some constant $D_3 < \infty$, then for every $M \geq D_2 D_3$ the GSLLN

$$\frac{\sum_{k=1}^{n} a_k (X_k - \mu_k)}{b_n} \to 0 \quad \text{a.s.} \tag{15}$$

obtains where $\mu_1 = EX_1 I(|X_1| \leq Mc_1)$ and $\mu_n = E\{X_n I(|X_n| \leq Mc_n)|X_1, \ldots, X_{n-1}\}$, $n \geq 2$. Furthermore, if

$$c_n = O(\inf_{k \geq n} c_k) \tag{16}$$

and

$$c_n \sum_{k=1}^{n} \frac{1}{c_k} = O(n) \tag{17}$$

then

$$\frac{\sum_{k=1}^{n} a_k(X_k - \nu_k)}{b_n} \to 0 \quad \text{a.s.} \tag{18}$$

where $\nu_1 = EX_1$ and $\nu_n = E\{X_n | X_1, \ldots, X_{n-1}\}$, $n \geq 2$.

Proof. Let $M \geq D_2 D_3$. The hypotheses ensure that (3) and (6) hold with $D = \max\{D_1, D_2, D_3\}$, whence by Lemma 3.2, (7) obtains with $q = 2$. Let $Z_n = X_n I(|X_n| \leq Mc_n)$, $n \geq 1$. Define $X_0 = 0$ and $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$, $n \geq 0$. Observe that for $k \geq 1$, μ_k is $\mathcal{F}_{k-1}$-measurable and hence

$$E\{\frac{a_k}{b_k}(Z_k - \mu_k) | \mathcal{F}_{k-1}\} = \frac{a_k}{b_k}(\mu_k - \mu_k) = 0 \quad \text{a.s.}$$

Thus, $\{\frac{a_n}{b_n}(Z_n - \mu_n), \mathcal{F}_n, n \geq 1\}$ is a martingale difference sequence. This is equivalent to saying that $\{\sum_{k=1}^{n} \frac{a_k}{b_k}(Z_k - \mu_k), \mathcal{F}_n, n \geq 1\}$ is a martingale.

It will now be shown that for $i < j$

$$E(Z_i - \mu_i)(Z_j - \mu_j) = 0. \tag{19}$$

Observe that

$$E(Z_i-\mu_i)(Z_j-\mu_j) = E\{E\{(Z_i-\mu_i)(Z_j-\mu_j)|\mathcal{F}_{j-1}\}\}$$

$$= E\{(Z_i-\mu_i)E\{Z_j-\mu_j|\mathcal{F}_{j-1}\}\} \qquad \text{(since } Z_i-\mu_i \text{ is } \mathcal{F}_{j-1}\text{-measurable)}$$

$$= 0.$$

Next, it will be shown that

$$\sum_{k=1}^{n} \frac{a_k}{b_k}(Z_k-\mu_k) \quad \text{converges a.s.} \tag{20}$$

In view of the martingale convergence theorem (see, e.g., Breiman, 1968, p. 89) it suffices to show that

$$\limsup_{n\to\infty} E\left|\sum_{k=1}^{n} \frac{a_k}{b_k}(Z_k-\mu_k)\right| < \infty. \tag{21}$$

Note that

$$E\{(Z_k-\mu_k)^2|\mathcal{F}_{k-1}\}$$

$$= E\{Z_k^2|\mathcal{F}_{k-1}\} - 2\mu_k E\{Z_k|\mathcal{F}_{k-1}\} + \mu_k^2 \qquad \text{(since } \mu_k \text{ is } \mathcal{F}_{k-1}\text{-measurable)}$$

$$= E\{Z_k^2|\mathcal{F}_{k-1}\} - \mu_k^2$$

$$\le E\{Z_k^2|\mathcal{F}_{k-1}\} \qquad \text{a.s.}$$

Thus

$$\limsup_{n\to\infty}\Big(E\Big|\sum_{k=1}^{n}\frac{a_k}{b_k}(Z_k-\mu_k)\Big|\Big)^2$$

$$\leq \sup_{n\geq 1} E\Big(\sum_{k=1}^{n}\frac{a_k}{b_k}(Z_k-\mu_k)\Big)^2$$

$$= \sup_{n\geq 1}\Big\{E\sum_{k=1}^{n}\Big(\frac{a_k}{b_k}\Big)^2(Z_k-\mu_k)^2 + 2E\sum_{i<j}^{n}\sum^{n}\Big(\frac{a_i}{b_i}\Big)\Big(\frac{a_j}{b_j}\Big)(Z_i-\mu_i)(Z_j-\mu_j)\Big\}$$

$$= \sup_{n\geq 1}\Big\{\sum_{k=1}^{n}\frac{1}{c_k^2}E(Z_k-\mu_k)^2 + 2\sum_{i<j}^{n}\sum^{n}\Big(\frac{a_i a_j}{b_i b_j}\Big)E(Z_i-\mu_i)(Z_j-\mu_j)\Big\}$$

$$= \sum_{k=1}^{\infty}\frac{1}{c_k^2}E(Z_k-\mu_k)^2 \qquad \text{(by (19))}$$

$$= \sum_{k=1}^{\infty}\frac{1}{c_k^2}E\{E\{(Z_k-\mu_k)^2|\mathfrak{F}_{k-1}\}\}$$

$$\leq \sum_{k=1}\frac{}{c_k^2}E\{E\{Z_k^2|\mathfrak{F}_{k-1}\}\}$$

$$= \sum_{k=1}^{\infty}\frac{1}{c_k^2}EZ_k^2$$

$$= \sum_{k=1}^{\infty} \frac{1}{c_k^2} EX_k^2 I(|X_k| \leq Mc_k) < \infty$$

as already noted, whence (21) obtains thereby establishing (20). Using $b_n \uparrow \infty$ and the Kronecker lemma, we conclude

$$\frac{\sum_{k=1}^{n} a_k(Z_k-\mu_k)}{b_n} \to 0 \quad \text{a.s.}$$

However,

$$\frac{\sum_{k=1}^{n} a_k(X_k-\mu_k)}{b_n} = \frac{\sum_{k=1}^{n} a_k(Z_k-\mu_k)}{b_n} + \frac{\sum_{k=1}^{n} a_k(X_k-Z_k)}{b_n},$$

and the second term is o(1) a.s. via $b_n \to \infty$ and the Borel-Cantelli lemma since

$$\sum_{n=1}^{\infty} P\{X_n \neq Z_n\} = \sum_{n=1}^{\infty} P\{|X_n| > Mc_n\}$$

$$\leq \sum_{n=1}^{\infty} P\{|X_n| > D_2D_3c_n\} \qquad \text{(since } M \geq D_2D_3\text{)}$$

$$\leq D_1 \sum_{n=1}^{\infty} P\{|X| > D_3c_n\} < \infty \qquad \text{(by (13) and (14)).}$$

Hence (15) obtains.

Now under (17), $c_n \leq Cn$ for all $n \geq 1$ whence for all $k \geq 1$

$$\sum_{n=1}^{\infty} P\{|X_k| > CD_2D_3n\} \leq D_1 \sum_{n=1}^{\infty} P\{|X| > CD_3n\} \qquad \text{(by (13))}$$

$$\leq D_1 \sum_{n=1}^{\infty} P\{|X| > D_3c_n\} < \infty \qquad \text{(by (14))}.$$

Therefore, via (11), we obtain $E|X| < \infty$ and $E|X_k| < \infty$, $k \geq 1$. This in turn guarantees that the conditional expectations all exist.

Let $e_n = \inf_{k \geq n} c_k$, $n \geq 1$. Note that, via (5), $e_n \uparrow \infty$ and $e_n \leq c_n$, $n \geq 1$. Under (16), $e_n \leq c_n \leq Ae_n$ for some $1 \leq A < \infty$.

Let $N \geq MA$. Utilizing Lemma 2.4 we obtain

$$E|X_n|I(|X_n| > Nc_n)$$

$$= Nc_nP\{|X_n| > Nc_n\} + \int_{Nc_n}^{\infty} P\{|X_n| > t\}dt$$

$$\leq D_1Nc_nP\{|X| > Nc_n/D_2\} + D_1\int_{Nc_n}^{\infty} P\{|X| > t/D_2\}dt \qquad \text{(by (13))}$$

$$= D_1[Nc_nP\{|D_2X| > Nc_n\} + \int_{Nc_n}^{\infty} P\{|D_2X| > t\}dt]$$

$$= D_1E|D_2X|I(|D_2X| > Nc_n).$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{c_n} E|X_n| I(|X_n| > Nc_n)$$

$$\leq D_1 \sum_{n=1}^{\infty} \frac{1}{c_n} E|D_2X| I(|D_2X| > Nc_n)$$

$$\leq D_1 \sum_{n=1}^{\infty} \frac{1}{c_n} E|D_2X| I(|D_2X| > Ne_n)$$

$$= D_1 \sum_{n=1}^{\infty} \frac{1}{c_n} \sum_{k=n}^{\infty} E|D_2X| I(Ne_k < |D_2X| \leq Ne_{k+1}) \qquad (\text{since } e_n \uparrow \infty)$$

$$= D_1 \sum_{k=1}^{\infty} E|D_2X| I(Ne_k < |D_2X| \leq Ne_{k+1}) \sum_{n=1}^{k} \frac{1}{c_n}$$

$$\leq D_1 \sum_{k=1}^{\infty} Ne_{k+1} P\{Ne_k < |D_2X| \leq Ne_{k+1}\} \sum_{n=1}^{k+1} \frac{1}{c_n}$$

$$\leq D_1 N \sum_{k=1}^{\infty} P\{Ne_k < |D_2X| \leq Ne_{k+1}\} c_{k+1} \sum_{n=1}^{k+1} \frac{1}{c_n} \qquad (\text{since } e_k \leq c_k)$$

$$\leq D_1 N \sum_{k=1}^{\infty} P\{Ne_k < |D_2X| \leq Ne_{k+1}\} C(k+1) \qquad (\text{by } (17))$$

$$= D_1 NC \sum_{k=1}^{\infty} (k+1) P\{Ne_k < |D_2X| \leq Ne_{k+1}\}$$

$$\leq 2D_1 NC \sum_{k=1}^{\infty} kP\{Ne_k < |D_2X| \leq Ne_{k+1}\}$$

$$= C \sum_{k=1}^{\infty} \sum_{n=1}^{k} P\{Ne_k < |D_2X| \leq Ne_{k+1}\}$$

$$= C \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} P\{Ne_k < |D_2X| \leq Ne_{k+1}\}$$

$$= C \sum_{n=1}^{\infty} P\{|D_2X| > Ne_n\}$$

$$\leq C \sum_{n=1}^{\infty} P\{|D_2X| > Mc_n\} \qquad (\text{since } Ne_n \geq MAe_n \geq Mc_n)$$

$$\leq C \sum_{n=1}^{\infty} P\{|X| > D_3c_n\} < \infty \qquad (\text{by } M \geq D_2D_3 \text{ and } (14)).$$

Hence, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{c_n} E|X_n| I(|X_n| > Nc_n) < \infty. \tag{22}$$

Next, it will be shown that

$$\sum_{n=1}^{\infty} \frac{1}{c_n} E|X_n| I(Mc_n < |X_n| \leq Nc_n) < \infty. \tag{23}$$

Using (13) and the fact that $M \geq D_2D_3$ we obtain

$$\sum_{n=1}^{\infty} \frac{1}{c_n} E|X_n| I(Mc_n < |X_n| \leq Nc_n)$$

$$\leq N \sum_{n=1}^{\infty} P\{Mc_n < |X_n| \leq Nc_n\}$$

$$\leq N \sum_{n=1}^{\infty} P\{|X_n| > Mc_n\}$$

$$\leq ND_1 \sum_{n=1}^{\infty} P\{|X| > Mc_n/D_2\}$$

$$\leq ND_1 \sum_{n=1}^{\infty} P\{|X| > D_3c_n\} < \infty \qquad \text{(by (14))}$$

establishing (23). Combining (22) and (23) yields

$$\sum_{n=1}^{\infty} \frac{1}{c_n} E|X_n| I(|X_n| > Mc_n)$$

$$= \sum_{n=1}^{\infty} \frac{1}{c_n} E|X_n| I(Mc_n < |X_n| \leq Nc_n) + \sum_{n=1}^{\infty} \frac{1}{c_n} E|X_n| I(|X_n| > Nc_n) < \infty.$$

Hence by the Beppo-Levi theorem,

$$E \sum_{n=1}^{\infty} \frac{1}{c_n} E\{|X_n| I(|X_n| > Mc_n) | \mathcal{F}_{n-1}\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{c_n} E\{E\{|X_n| I(|X_n| > Mc_n) | \mathcal{F}_{n-1}\}\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{c_n} E|X_n| I(|X_n| > Mc_n) < \infty.$$

This implies that

$$\sum_{n=1}^{\infty} \frac{|a_n|}{b_n} E\{|X_n| I(|X_n| > Mc_n) | \mathcal{F}_{n-1}\} < \infty \quad \text{a.s.}$$

Then by Jensen's inequality for conditional expectations and the Kronecker lemma we conclude that

$$\frac{|\sum_{k=1}^{n} a_k(\mu_k - \nu_k)|}{b_n}$$

$$= \frac{|\sum_{k=1}^{n} a_k(E\{X_k I(|X_k| \le Mc_k) | \mathcal{F}_{k-1}\} - E\{X_k | \mathcal{F}_{k-1}\})|}{b_n}$$

$$= \frac{|\sum_{k=1}^{n} a_k E\{X_k I(|X_k| > Mc_k) | \mathcal{F}_{k-1}\}|}{b_n}$$

$$\le \frac{\sum_{k=1}^{n} |a_k| E\{|X_k| I(|X_k| > Mc_k) | \mathcal{F}_{k-1}\}}{b_n} \to 0 \quad \text{a.s.}$$

Therefore, recalling (15),

$$\frac{\sum_{k=1}^{n} a_k(X_k-\nu_k)}{b_n} = \frac{\sum_{k=1}^{n} a_k(X_k-\mu_k)}{b_n} + \frac{\sum_{k=1}^{n} a_k(\mu_k-\nu_k)}{b_n} \to 0 \quad \text{a.s.}$$

thereby proving (18).□

Remark. Note that condition (16) is automatic if $c_n \uparrow$. Furthermore, if $c_n/n^\alpha \uparrow$ for some $\alpha > 0$ then, via Lemma 3.6,

$$\sum_{n=1}^{\infty} P\{|X| > \lambda c_n\} < \infty \qquad \text{for all } \lambda > 0.$$

Consequently, $D_2\lambda$ and hence M can be chosen arbitrarily small and so (15) holds for all $M > 0$.

This first corollary is a well known SLLN for sums of i.i.d. random variables and is essentially due to Feller (1946). It is an extension of the Marcinkiewicz-Zygmund SLLN to more general norming constants.

Corollary 3.1 (Feller, 1946). Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables and let $\{b_n, n \geq 1\}$ be positive constants. Suppose that either

(i) $EX = 0$, $n = O(b_n)$

or

(ii) $EX = 0$, $b_n/n \downarrow$, $b_n/n^\alpha \uparrow$ for some $\alpha > \frac{1}{2}$

or

(iii) $E|X| = \infty$, $b_n/n \uparrow$.

If

$$\sum_{n=1}^{\infty} P\{|X| > b_n\} < \infty, \tag{24}$$

then

$$\frac{\sum_{k=1}^{n} X_k}{b_n} \to 0 \quad \text{a.s.} \tag{25}$$

Proof. Clearly (13) holds with $D_1 = D_2 = 1$ and $X = X_1$. Defining $a_n = 1$, $n \geq 1$, we obtain, via (2), that $c_n = b_n$, $n \geq 1$. Thus (24) is equivalent to (14) where $D_3 \equiv 1$.

Suppose (ii) or (iii) hold. Then $b_n \uparrow \infty$ and $b_n/n^\beta \uparrow$ for some $\beta > 1/2$. Therefore

$$\frac{b_{rn}/(rn)^\beta}{b_n/n^\beta} \geq 1$$

for all integers $r \geq 2$. Hence, in particular, $\frac{b_{2n}}{b_n} \geq 2^\beta > 2^{1/2}$ and so

$$\liminf_{n\to\infty} \frac{b_{2n}}{b_n} > 2^{1/2}.$$

This together with $b_n^2/n \uparrow$ shows that, via Lemma 3.5,

$$b_n^2 \sum_{k=n}^{\infty} \frac{1}{b_k^2} = O(n).$$

Utilizing Theorem 3.1 we conclude that

$$\frac{\sum_{k=1}^{n}(X_k-\mu_k)}{b_n} \to 0 \quad \text{a.s.} \tag{26}$$

where $\mu_n = EXI(|X| \leq b_n)$, $n \geq 1$. It remains to show that

$$\sum_{k=1}^{n}\mu_k = o(b_n) \tag{27}$$

in each of the cases (ii) and (iii).

In case (ii), we have $EX = 0$, $b_n/n \downarrow$, $b_n^2 \sum_{k=n}^{\infty} \frac{1}{b_k^2} = O(n)$, and $\sum_{n=1}^{\infty} P\{|X| > b_n\} < \infty$ which are sufficient for (27) (see Chow and Teicher, 1978, p. 124).

In case (iii), note that $b_n/n \uparrow \infty$, $b_n^2 \sum_{k=n}^{\infty} \frac{1}{b_k^2} = O(n)$, and $\sum_{n=1}^{\infty} P\{|X| > b_n\} < \infty$ all hold. As in the previous case these assumptions imply (27) (again see Chow and Teicher, 1978, p. 124).

Therefore in either case (ii) or (iii), both (26) and (27) obtain and then (25) follows.

Next, suppose (i) holds. Then by applying case (ii) with the norming constants $b_n' = n$, $n \geq 1$, it follows that

$$\frac{\sum_{k=1}^{n}X_k}{n} \to 0 \quad \text{a.s.}$$

Since $n = O(b_n)$

$$\frac{\sum_{k=1}^{n} X_k}{b_n} = \frac{n}{b_n} \cdot \frac{\sum_{k=1}^{n} X_k}{n} \to 0 \quad \text{a.s.}$$

thereby proving case (i) and the corollary. □

Remark. It will now be shown that under case (ii) the condition (24) need not be assumed, but rather is automatic.

Fact. If $\{X, X_n, n \geq 1\}$ are i.i.d. random variables and $\{b_n, n \geq 1\}$ are positive constants with $EX = 0$ and $n = O(b_n)$, then (24) holds.

Proof. Since $n = O(b_n)$, there is some finite constant C such that $n \leq Cb_n$. Utilizing (11), with $p = 1$, we obtain

$$\sum_{n=1}^{\infty} P\{|CX| > n\} < \infty.$$

Therefore

$$\sum_{n=1}^{\infty} P\{|X| > b_n\} \leq \sum_{n=1}^{\infty} P\{|CX| > n\} < \infty. \square$$

As pointed out previously, Corollary 3.1 is a generalization of the Marcinkiewicz-Zygmund SLLN. The Marcinkiewicz-Zygmund SLLN is, in turn, a generalization of the famous Kolmogorov SLLN (see, e.g., Chow and Teicher, 1978, p. 122). The next corollary extends Theorem 5.2.3(ii) of Chow and Teicher (1978, p. 123).

Corollary 3.2. Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables with $EX = 0$. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be constants satisfying $0 < b_n \uparrow \infty$,

$$\sum_{k=1}^{n} |a_k| = O(b_n), \tag{28}$$

and (5) with $q = 2$. Then

$$\frac{\sum_{k=1}^{n} a_k X_k}{b_n} \to 0 \quad \text{a.s.} \tag{29}$$

iff

$$\sum_{n=1}^{\infty} P\{|X| > c_n\} < \infty \tag{30}$$

where c_n is as in (2).

Proof. Again, let $D_1 = D_2 = D_3 = 1$ and hence (13) and (14) hold. Thus, via Theorem 3.1, (15) obtains with $\mu_n = EXI(|X| \leq c_n)$, $n \geq 1$, and so if

$$\sum_{k=1}^{n} a_k \mu_k = o(b_n) \tag{31}$$

then (29) follows. To prove (31), note that (5) ensures that $c_n \to \infty$, implying by the Lebesgue dominated convergence theorem that $\mu_n \to EX = 0$. Then (31) follows immediately from (28) and the Toeplitz lemma.

Conversely, if (29) holds, then for all $n \geq 2$

$$\left|\frac{X_n}{c_n}\right| = \frac{|a_n X_n|}{b_n} = \frac{\left|\sum_{k=1}^{n} a_k X_k - \sum_{k=1}^{n-1} a_k X_k\right|}{b_n}$$

$$\leq \frac{\left|\sum_{k=1}^{n} a_k X_k\right|}{b_n} + \frac{\left|\sum_{k=1}^{n-1} a_k X_k\right|}{b_{n-1}} \to 0 \quad \text{a.s.}$$

Therefore

$$P\{|X_n/c_n| > 1 \text{ i.o. } (n)\} = 0$$

and (30) follows by the Borel-Cantelli lemma. □

The similarities and differences between Theorem 3.1 and the next result are quite evident. It should be noted that, via Lemma 3.3, if $c_n \uparrow$ and (5) holds with $q = 1$ then (5) obtains with $q = 2$. In the special case where $\{X_n, n \geq 1\}$ are identically distributed and $a_n \equiv 1$, Theorem 3.2 reduces to Theorem 2 of Martikainen and Petrov (1980).

Theorem 3.2. (i) Let $\{X_n, n \geq 1\}$ and X be random variables such that $\{|X_n|, n \geq 1\}$ is stochastically dominated by $|X|$ in the sense that there exists a constant $D < \infty$ such that

$$P\{|X_n| > t\} \leq DP\{|X| > t/D\}, \; t \geq 0, \; n \geq 1. \tag{32}$$

Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be constants satisfying $0 < b_n \uparrow \infty$ and (5) with c_n as in (2) and $q = 1$. If

$$\sum_{n=1}^{\infty} P\{|X| > Dc_n\} < \infty, \tag{33}$$

then $\sum_{k=1}^{n} a_k X_k / b_n \to 0$ a.s.

(ii) Conversely, let $\{X_n, n \geq 1\}$ be pairwise independent random variables and let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be constants with $b_n > 0$ and $b_{n-1} = O(b_n)$. If $\sum_{k=1}^{\infty} a_k X_k / b_n \to 0$ a.s., then

$$\sum_{n=1}^{\infty} P\{|X_n| > \varepsilon c_n\} < \infty \quad \text{for all } \varepsilon > 0. \tag{34}$$

Proof. (i) By the Beppo-Levi theorem and Lemma 3.2,

$$E \sum_{k=1}^{\infty} \frac{1}{b_k} |a_k X_k| I(|X_k| \leq D^2 c_k)$$

$$= \sum_{k=1}^{\infty} \frac{1}{c_k} E|X_k| I(|X_k| \leq D^2 c_k) < \infty$$

whence

$$\sum_{k=1}^{\infty} \frac{1}{b_k} |a_k X_k| I(|X_k| \leq D^2 c_k) < \infty \quad \text{a.s.}$$

Then, by the Kronecker lemma, it follows that

$$\frac{\sum_{k=1}^{n} |a_k X_k| I(|X_k| \le D^2 c_k)}{b_n} \to 0 \quad \text{a.s.} \tag{35}$$

However, by (32) and (33)

$$\sum_{n=1}^{\infty} P\{X_n \ne X_n I(|X_n| \le D^2 c_n)\}$$

$$= \sum_{n=1}^{\infty} P\{|X_n| > D^2 c_n\} \le D \sum_{n=1}^{\infty} P\{|X| > Dc_n\} < \infty.$$

The Borel-Cantelli lemma then ensures that

$$P\{\liminf_{n\to\infty}[X_n = X_n I(|X_n| \le D^2 c_n)]\} = 1$$

implying, via $b_n \to \infty$, that

$$\frac{\sum_{k=1}^{n} |a_k X_k| I(|X_k| > D^2 c_k)}{b_n} \to 0 \quad \text{a.s.} \tag{36}$$

Combining (35) and (36) yields

$$\frac{|\sum_{k=1}^{n} a_k X_k|}{b_n} \le \frac{\sum_{k=1}^{n} |a_k X_k|}{b_n}$$

$$= \frac{\sum_{k=1}^{n} |a_k X_k| I(|X_k| \le D^2 c_k)}{b_n} + \frac{\sum_{k=1}^{n} |a_k X_k| I(|X_k| > D^2 c_k)}{b_n}$$

$$\to 0 \quad \text{a.s.}$$

proving part (i).

(ii) If (29) and $b_{n-1} = O(b_n)$ hold we obtain, for all $n \ge 2$,

$$\frac{|a_n X_n|}{b_n} = \frac{\left|\sum_{k=1}^{n} a_k X_k - \sum_{k=1}^{n-1} a_k X_k\right|}{b_n}$$

$$\le \frac{\left|\sum_{k=1}^{n} a_k X_k\right|}{b_n} + \frac{b_{n-1}}{b_n} \cdot \frac{\left|\sum_{k=1}^{n-1} a_k X_k\right|}{b_{n-1}} \to 0 \quad \text{a.s.}$$

Therefore $P\{|X_n| > \varepsilon c_n \text{ i.o. } (n)\} = 0$ for all $\varepsilon > 0$, and so (34) obtains by the Borel-Cantelli lemma for pairwise independent events (see, e.g., Chung, 1974, p. 76, or, for a new simple proof, Etemadi, 1984). □

Remark. If we replace conditions (32) and (33) of Theorem 3.2 with (13) and (14) then the conclusion of Theorem 3.2(i) still obtains.

Proof. Suppose that (13) and (14) hold. Let $D = \max\{D_1, D_2, D_3\}$. Then if (13) holds we obtain, for all $t \ge 0$ and $n \ge 1$,

$$P\{|X_n| > t\} \le D_1 P\{|X| > t/D_2\} \le DP\{|X| > t/D\},$$

which is (13). Next, if (14) is satisfied, then

$$\sum_{n=1}^{\infty} P\{|X| > Dc_n\} \le \sum_{n=1}^{\infty} P\{|X| > D_3 c_n\} < \infty,$$

whence (33) obtains. The remark then follows from Theorem 3.2. □

Theorem 3.2 produces a few immediate corollaries, the third of which is a result of Martikainen and Petrov (1980).

Corollary 3.3. Let $\{X, X_n, n \ge 1\}$ be pairwise i.i.d. (p.i.i.d.) random variables. Let $\{a_n, n \ge 1\}$ and $\{b_n, n \ge 1\}$ be constants with $0 < b_n \uparrow \infty$, $c_n \uparrow$ and

$$c_n \sum_{k=n}^{\infty} \frac{1}{c_k} = O(n). \tag{37}$$

Then $\sum_{k=1}^{n} a_k X_k / b_n \to 0$ a.s. iff (30) holds.

Proof. Suppose (30) holds. Then, by letting $D = 1$ and $X = X_1$ we see that (32) holds and (30) is tantamount to (33). Noting that $c_n \uparrow$, it is evident that (37) is commensurable to (5) when $q = 1$. Thus, all the hypotheses of Theorem 3.2(i) are satisfied, whence

$$\sum_{k=1}^{n} a_k X_k / b_n \to 0 \text{ a.s.}$$

For the converse, note that if $0 < b_n \uparrow \infty$ then $b_{n-1} = O(b_n)$. Then if $\sum_{k=1}^{n} a_k X_k / b_n \to 0$ a.s., we can conclude (34) with $\varepsilon = 1$ which is tantamount to (30). □

Corollary 3.4. Let $\{X, X_n, n \geq 1\}$ are p.i.i.d. random variables. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be constants where $0 < |a_n| \downarrow$, $0 < b_n \uparrow$, and

$$b_n \sum_{k=n}^{\infty} \frac{1}{b_k} = O(n), \tag{38}$$

then $\sum_{k=1}^{n} a_k X_k / b_n \to 0$ a.s. iff (30) holds.

Proof. Since $b_n \uparrow$ and $|a_n| \downarrow$, we obtain $c_n \uparrow$. Since $b_n \uparrow \infty$ (by (38)), in view of Corollary 3.3, it remains to show that (37) holds. However, this is immediate since, via Lemma 3.4, $0 < b_n \uparrow \infty$, $0 < |a_n| \downarrow$, and (38) ensure (37). □

Corollary 3.5 (Martikainen and Petrov, 1980, Theorem 2). (i) Let $\{X, X_n, n \geq 1\}$ be identically distributed random variables. Let $\{b_n, n \geq 1\}$ be a sequence of constants satisfying $0 < b_n \uparrow \infty$ and (38). If

$$\sum_{n=1}^{\infty} P\{|X| > b_n\} < \infty \tag{39}$$

then

$$\frac{\sum_{k=1}^{n} X_k}{b_n} \to 0 \quad \text{a.s.} \tag{40}$$

(ii) Conversely, if the random variables $\{X, X_n, n \geq 1\}$ are p.i.i.d. and $\{b_n, n \geq 1\}$ is a sequence of positive constants with $b_n \uparrow \infty$, then (40) implies (39).

Proof. (i) Utilize Theorem 3.2(i) with $D = 1$, $X = X_1$, and $a_n \equiv 1$. Hence (32) and (33) are satisfied since $c_n = b_n$, $n \geq 1$. Then (38) is tantamount to (5) with $q = 1$, whence (40) follows.

(ii) This follows immediately from Theorem 3.2(ii). □

3.4 Generalized Strong Laws for Large Numbers for Weighted Sums of Mean Zero Random Variables

The previous section was devoted to GSLLNs for stochastically dominated random variables. This section will focus on the GSLLN problem when the random variables are i.i.d. with mean zero. Since the hypotheses imposed on the distributions of the random variables $\{X_n, n \geq 1\}$ are being strengthened, our goal will be to weaken the assumptions placed on the sequences $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$.

The first theorem of this section was motivated by Fernholz and Teicher (1980) and produces some of their results as corollaries.

Theorem 3.3. Let $\{X, X_n, n \geq 1\}$ be i.i.d. $\mathcal{L}_p$ random variables for some $1 \leq p < 2$ where $EX = 0$. If $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ are sequences of constants with $0 < b_n \uparrow \infty$,

$$a_n/b_n = O(n^{\frac{p-3}{2}}), \qquad (41)$$

and

$$\sum_{k=1}^{n} |a_k| = O(b_n) \qquad (42)$$

then

$$\frac{\sum_{k=1}^{n} a_k X_k}{b_n} \to 0 \quad \text{a.s.} \tag{43}$$

Proof. Let $Z_n = X_n I(|X_n| \le n)$, $n \ge 1$, and $W_n = \frac{a_n}{b_n}(Z_n - EZ_n)$, $n \ge 1$. It will be shown that the series

$$\sum_{n=1}^{\infty} W_n \quad \text{converges a.s.} \tag{44}$$

By the Khintchine-Kolmogorov convergence theorem (see, e.g., Chow and Teicher, 1978, p. 110) we only need to show that $\sum_{n=1}^{\infty} EW_n^2 < \infty$. Now

$$\sum_{n=1}^{\infty} EW_n^2 = \sum_{n=1}^{\infty} \left(\frac{a_n}{b_n}\right)^2 E(Z_n - EZ_n)^2$$

$$\le \sum_{n=1}^{\infty} \left(\frac{a_n}{b_n}\right)^2 E(Z_n^2) = \sum_{n=1}^{\infty} \left(\frac{a_n}{b_n}\right)^2 EX^2 I(|X| \le n)$$

$$= \sum_{n=1}^{\infty} \left(\frac{a_n}{b_n}\right)^2 \sum_{k=1}^{n} EX^2 I(k-1 < |X| \le k)$$

$$= \sum_{k=1}^{\infty} EX^2 I(k-1 < |X| \le k) \sum_{n=k}^{\infty} \left(\frac{a_n}{b_n}\right)^2$$

$$\leq \sum_{k=1}^{\infty} EX^2 I(k-1 < |X| \leq k) C \sum_{n=k}^{\infty} n^{p-3} \qquad \text{(by (41))}$$

$$\leq C \sum_{k=1}^{\infty} k^{p-2} EX^2 I(k-1 < |X| \leq k)$$

$$\leq C \sum_{k=1}^{\infty} E|X|^p \left|\frac{X}{k}\right|^{2-p} I(k-1 < |X| \leq k)$$

$$\leq C \sum_{k=1}^{\infty} E|X|^p I(k-1 < |X| \leq k)$$

$$= CE|X|^p < \infty,$$

whence (44) obtains which is tantamount to

$$\sum_{n=1}^{\infty} \frac{a_n}{b_n}(Z_n - EZ_n) \quad \text{converges a.s.}$$

Next, via the Kronecker lemma we obtain

$$\frac{\sum_{k=1}^{n} a_k(Z_k - EZ_k)}{b_n} \to 0 \quad \text{a.s.}$$

However,

$$\sum_{n=1}^{\infty} P\{Z_n \neq X_n\} = \sum_{n=1}^{\infty} P\{|X_n| > n\} < \infty \qquad \text{(by (11))}$$

and so by the Borel-Cantelli lemma

$$P\{\liminf_{n\to\infty}[Z_n = X_n]\} = 1.$$

Hence

$$\frac{\sum_{k=1}^{n} a_k(X_k - EZ_k)}{b_n} = \frac{\sum_{k=1}^{n} a_k(X_k - Z_k)}{b_n} + \frac{\sum_{k=1}^{n} a_k(Z_k - EZ_k)}{b_n} \to 0 \quad \text{a.s.} \tag{45}$$

Next, we show that the sequence

$$\frac{\sum_{k=1}^{n} a_k EZ_k}{b_n} = o(1). \tag{46}$$

Let $\varepsilon > 0$. In view of (42) there exists a constant $C < \infty$ such that

$$\frac{\sum_{k=1}^{n} |a_k|}{b_n} \leq C, \qquad n \geq 1. \tag{47}$$

Noting that $E|X| < \infty$ we realize that there exists an integer n_o such that whenever $k \geq n_o$

$$E|X|I(|X| > k) \leq \frac{\varepsilon}{C}. \tag{48}$$

Thus, for large n

$$\frac{\left|\sum_{k=1}^{n} a_k EZ_k\right|}{b_n} \le \frac{1}{b_n} \sum_{k=1}^{n} |a_k| \, |EXI(|X| \le k)|$$

$$= \frac{1}{b_n} \sum_{k=1}^{n} |a_k| \, |EXI(|X| > k)| \qquad \text{(since } EX = 0\text{)}$$

$$\le \frac{1}{b_n} \sum_{k=1}^{n_o} |a_k| E|X| I(|X| > k) + \frac{1}{b_n} \sum_{k=n_o+1}^{n} |a_k| E|X| I(|X| > k)$$

$$\le \frac{1}{b_n} \sum_{k=1}^{n_o} |a_k| E|X| + \frac{1}{b_n} \sum_{k=n_o+1}^{n} |a_k| \left(\frac{\varepsilon}{C}\right) \qquad \text{(by (48))}$$

$$\le o(1) + \varepsilon \qquad \text{(by } b_n \to \infty \text{ and (47))}$$

$$\to \varepsilon \quad \text{as } n \to \infty.$$

Since $\varepsilon > 0$ is arbitrary (46) follows. This together with (45) yields

$$\frac{\sum_{k=1}^{n} a_k X_k}{b_n} = \frac{\sum_{k=1}^{n} a_k (X_k - EZ_k)}{b_n} + \frac{\sum_{k=1}^{n} a_k EZ_k}{b_n} \to 0 \quad \text{a.s.}$$

thereby proving the theorem. □

Remark. It is quite evident that Theorem 3.3 resembles Corollary 3.2. The question that arises is whether or not one of these results implies the other. It will be shown that these results indeed stand on their own although some of their hypotheses do imply those of the other.

Fact. If $X \in \mathcal{L}_p$ and $a_n/b_n = O(n^{\frac{p-3}{2}})$ for some $1 \leq p < 2$, then $\sum_{n=1}^{\infty} P\{|X| > c_n\} < \infty$.

Proof. Recall that $c_n = b_n/|a_n|$ whence $\frac{1}{c_n} \leq Cn^{\frac{p-3}{2}}$, $n \geq 1$, or $c_n \geq \varepsilon n^{\frac{3-p}{2}}$ for some $\varepsilon > 0$ and all $n \geq 1$. Note that since $1 \leq p < 2$, $\frac{(3-p)p}{2} \geq 1$. Hence

$$\sum_{n=1}^{\infty} P\{|X| > c_n\} \leq \sum_{n=1}^{\infty} P\{|X| > \varepsilon n^{\frac{3-p}{2}}\}$$

$$= \sum_{n=1}^{\infty} P\{|\tfrac{X}{\varepsilon}|^p > n^{\frac{(3-p)p}{2}}\}$$

$$\leq \sum_{n=1}^{\infty} P\{|\tfrac{X}{\varepsilon}|^p > n\} < \infty \qquad \text{(by (11))}. \ \square$$

Next, it will be shown that neither result implies the other.

Example 3.1. Let $\{X, X_n, n \geq 1\}$ be i.i.d. mean zero random variables. Suppose that $E|X|^{3/2} < \infty$, but $E|X|^p = \infty$ for all

$p > 3/2$. If $a_n = n^{-2/3}e^n$, $n \geq 1$, and $b_n = e^n$, $n \geq 1$, then the hypotheses (28), (5) (with $q = 2$), and (30) of Corollary 3.2 hold, but the hypotheses of Theorem 3.3 fail.

Proof. Firstly, we verify (28) (which is the same as (42)). Observe that

$$\sum_{k=1}^{n} |a_k| = \sum_{k=1}^{n} k^{-2/3}e^k \leq \sum_{k=1}^{n} e^k \leq \int_1^{n+1} e^x dx$$

$$\leq e^{n+1} = O(b_n).$$

Next, we establish (5) (with $q = 2$) and (30). Note that $c_n = n^{2/3}$ and hence

$$\max_{1 \leq k \leq n} c_k^2 \sum_{j=n}^{\infty} \frac{1}{c_j^2} = n^{4/3} \sum_{j=n}^{\infty} \frac{1}{j^{4/3}} = n^{4/3}O(n^{-1/3}) = O(n)$$

which establishes (5) with $q = 2$. To show that (30) holds, note via (11) that

$$\sum_{n=1}^{\infty} P\{|X| > c_n\} = \sum_{n=1}^{\infty} P\{|X| > n^{2/3}\} < \infty$$

since $E|X|^{3/2} < \infty$.

Next, we show that the hypotheses of Theorem 3.3 fail. In this example, $X \in \mathcal{L}_p$ iff $p \leq 3/2$ which is equivalent to $\frac{3-p}{2} \geq 3/4$. Also recall that (41) is tantamount to $c_n \geq \varepsilon n^{\frac{3-p}{2}}$ for some $\varepsilon > 0$ and all $n \geq 1$. Hence if $X \in \mathcal{L}_p$ for some $1 \leq p \leq 3/2$ and if (41) holds, then $c_n \geq \varepsilon n^{3/4}$ for some $\varepsilon > 0$ and all $n \geq 1$ which contradicts $c_n = n^{2/3}$, $n \geq 1$. Thus the hypotheses of Theorem 3.3 cannot be satisfied. □

Example 3.2. Let $\{X, X_n, n \geq 1\}$ be i.i.d. mean zero random variables. Let

$$a_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ \frac{1}{n} & \text{if } n \text{ is even,} \end{cases}$$

and $b_n = n$, $n \geq 1$. Then the hypotheses of Theorem 3.3 hold, but the hypothesis (5) (with $q = 2$) of Corollary 3.2 fails.

Proof. Clearly (41) holds with $p = 1$. Likewise (42) obtains since $\sum_{k=1}^{n} |a_k| \leq n = O(b_n)$. Note that

$$c_n = \begin{cases} n & \text{if } n \text{ is odd,} \\ n^2 & \text{if } n \text{ is even.} \end{cases}$$

To show that (5) fails when $q = 2$, let

$$d_j = \begin{cases} \dfrac{1}{j^2} & \text{if } j \text{ is odd,} \\ 0 & \text{if } j \text{ is even} \end{cases}$$

and $e_j = \frac{1}{j^3}$, $j \geq 1$. Also, let n_o satisfy

$$j^3 \geq 2(j+1)^2 \qquad \text{for } j \geq n_o.$$

Now, if $j \geq n_o$, then

$$d_j + d_{j+1} \geq \frac{1}{(j+1)^2} \geq \frac{2}{j^3} \geq \frac{1}{j^3} + \frac{1}{(j+1)^3} = e_j + e_{j+1}.$$

Therefore $\sum_{j=n}^{\infty} d_j \geq \sum_{j=n}^{\infty} e_j$ obtains for all $n \geq n_o$. Thus, for $n \geq n_o$

$$\max_{1 \leq k \leq n} c_k^2 \sum_{j=n}^{\infty} \frac{1}{c_j^2} \geq (n-1)^4 \sum_{j=n}^{\infty} d_j$$

$$\geq (n-1)^4 \sum_{j=n}^{\infty} e_j = (n-1)^4 \sum_{j=n}^{\infty} \frac{1}{j^3}$$

$$\geq \frac{(n-1)^4}{2n^2} = (1+o(1))\frac{n^2}{2} \neq O(n),$$

which shows that (5) fails when $q = 2$. $\square$

These two examples verify the assertion that these results stand on their own. From Theorem 3.3 we obtain a few immediate corollaries.

Corollary 3.6. Let $\{X, X_n, n \geq 1\}$ be i.i.d. $\mathcal{L}_p$ random variables for some $1 \leq p < 2$ where $EX = 0$. If $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ are constants where $|a_n| \downarrow$, $0 < b_n \uparrow \infty$, and (42) holds, then (43) obtains.

Proof. In view of Theorem 3.3 we need only show that (41) is satisfied. Since $|a_n| \downarrow$, we have

$$n^{\frac{3-p}{2}} \frac{|a_n|}{b_n} = n^{\frac{1-p}{2}} \frac{n|a_n|}{b_n}$$

$$\leq n^{\frac{1-p}{2}} \frac{\sum_{k=1}^{n} |a_k|}{b_n} = n^{\frac{1-p}{2}} O(1) \qquad \text{(by (42))}$$

$$= o(1) \qquad \text{(since } 1 \leq p < 2\text{)},$$

which establishes (41). □

This next corollary was obtained by Fernholz and Teicher (1980).

Corollary 3.7 (Fernholz and Teicher, 1980, p. 769). Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables with $EX = 0$ and let $\{a_n, n \geq 1\}$ be positive constants. Set $A_n = \sum_{k=1}^{n} a_k$, $n \geq 1$. If for some $\beta \geq 0$

$$\frac{na_n}{A_n} = O((\log_2 A_n)^{\beta}), \ A_n \to \infty \tag{49}$$

then

$$\frac{\sum_{k=1}^{n} a_k X_k}{A_n(\log_2 A_n)^{\beta}} \to 0 \quad \text{a.s.} \tag{50}$$

Proof. Let $b_n = A_n(\log_2 A_n)^{\beta}$, $n \geq 1$, and $p = 1$. Hence, via Theorem 3.3, we only need to verify that $\frac{a_n}{b_n} = O(n^{-1})$ and $\sum_{k=1}^{n} a_k = O(b_n)$.

Observe that

$$\frac{na_n}{b_n} = \frac{na_n}{A_n(\log_2 A_n)^{\beta}} = O(1) \qquad \text{(by (49))}$$

and

$$\frac{\sum_{k=1}^{n} a_k}{b_n} = \frac{A_n}{A_n(\log_2 A_n)^{\beta}} = \frac{1}{(\log_2 A_n)^{\beta}} = O(1)$$

since $\beta \geq 0$. ☐

Clearly we can weaken the hypothesis (49) of Corollary 3.7 and still obtain a strong law (but with faster growing norming constants).

In Corollary 3.6 we realized that if we assume $|a_n| \downarrow$ then (42) implies (41) for any $1 \leq p < 2$. This next corollary looks at the reverse implication. It shows that if $p = 1$ and $|a_n| \uparrow$ then (41) implies (42).

Corollary 3.8. Let $\{X, X_n, n \geq 1\}$ be i.i.d. mean zero random variables. If $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ are constants such that $0 < b_n \uparrow \infty$, $|a_n| \uparrow$, and

$$na_n = O(b_n) \tag{51}$$

then (43) holds.

Proof. Apply Theorem 3.3 with $p = 1$ and note that (41) reduces to (51). Thus, we need only verify (42). Since $|a_n| \uparrow$

$$\frac{\sum_{k=1}^{n} |a_k|}{b_n} \leq \frac{n|a_n|}{b_n} = O(1) \qquad \text{(by (51))}$$

establishing (42).☐

We conclude this section with a theorem that utilizes a result from Chapter 2. More specifically, we will use a fact about regularly varying functions in our quest for a strong law.

Theorem 3.4. Let $\{X, X_n, n \geq 1\}$ be i.i.d. mean zero random variables with $P\{|X| > x\}$ regularly varying with exponent $\rho < -1$. Let

$\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be constants such that $0 < b_n \uparrow \infty$ and (5) holds with $q = 2$. Then

$$\frac{\sum_{k=1}^{n} a_k X_k}{b_n} \to 0 \quad \text{a.s.}$$

iff

$$\sum_{n=1}^{\infty} P\{|X| > c_n\} < \infty.$$

Proof. The necessity direction is immediate. For note that if $n \geq 2$,

$$\frac{|a_n X_n|}{b_n} = \frac{\left|\sum_{k=1}^{n} a_k X_k - \sum_{k=1}^{n-1} a_k X_k\right|}{b_n}$$

$$\leq \frac{\left|\sum_{k=1}^{n} a_k X_k\right|}{b_n} + \frac{\left|\sum_{k=1}^{n-1} a_k X_k\right|}{b_{n-1}} \to 0 \quad \text{a.s.}$$

Therefore $P\{\frac{|X_n|}{c_n} > 1 \text{ i.o. } (n)\} = 0$ whence $\sum_{n=1}^{\infty} P\{|X| > c_n\} < \infty$ follows from the Borel-Cantelli lemma.

To prove the sufficiency half, note by Theorem 3.1 and the independence hypothesis that

$$\frac{\sum_{k=1}^{n} a_k(X_k - EX_kI(|X_k| \le c_k))}{b_n} \to 0 \quad \text{a.s.} \tag{52}$$

Next, we will verify that

$$\sum_{k=1}^{\infty} \frac{a_k}{b_k} EX_kI(|X_k| \le c_k) \quad \text{converges.} \tag{53}$$

By Lemma 2.5, for all $k \ge$ some k_o

$$E|X|I(|X| > c_k) \le Cc_kP\{|X| > c_k\}. \tag{54}$$

Then

$$\sum_{k=1}^{\infty} \left|\frac{a_k}{b_k} EX_kI(|X_k| \le c_k)\right|$$

$$= \sum_{k=1}^{\infty} \frac{1}{c_k} |EXI(|X| \le c_k)|$$

$$= \sum_{k=1}^{\infty} \frac{1}{c_k} |EXI(|X| > c_k)| \qquad \text{(since } EX = 0\text{)}$$

$$\le \sum_{k=1}^{\infty} \frac{1}{c_k} E|X|I(|X| > c_k)$$

$$= \sum_{k=1}^{k_o} \frac{1}{c_k} E|X|I(|X| > c_k) + \sum_{k=k_o+1}^{\infty} \frac{1}{c_k} E|X|I(|X| > c_k)$$

$$\leq \sum_{k=1}^{k_0} \frac{1}{c_k} E|X| + C \sum_{k=k_0+1}^{\infty} P\{|X| > c_k\} \qquad \text{(by (54))}$$

$$< \infty \qquad \text{(by hypothesis)}$$

thereby proving (53). Then by the Kronecker lemma,

$$\frac{\sum_{k=1}^{n} a_k E X_k I(|X_k| \leq c_k)}{b_n} \to 0.$$

Combining this with (52) yields

$$\frac{\sum_{k=1}^{n} a_k X_k}{b_n} \to 0 \quad \text{a.s.} \quad \square$$

What will prove to be most interesting is an example that compares Theorem 3.4 and Theorem 2.2. We will establish a GCLT and a GSLLN for the same sequence of i.i.d. random variables with infinite second moment. This will be done in Section 3.6.

3.5 The Petersburg Game

Throughout this section, we will adopt the notation $S_n = \sum_{k=1}^{n} a_k X_k$, $n \geq 1$, where $\{a_n, n \geq 1\}$ are constants and the random variables $\{X, X_n, n \geq 1\}$ are i.i.d.

This section examines the Petersburg game. The idea behind this game is to find a sequence $\{m_n, n \geq 1\}$ so that $\sum_{k=1}^{n} X_k/m_n$ converges in some sense to 1. According to the (classical) Kolmogorov SLLN, if $\{X, X_n, n \geq 1\}$ are i.i.d. $\mathcal{L}_1$ random variables then $\sum_{k=1}^{n} X_k/n \to EX$ a.s.; the problem that arose was what could be done if $E|X| = \infty$.

This game occurs most naturally. The random variable X_n represents the winnings from the n^{th} play of a particular game. The sequence $\{m_n, n \geq 1\}$ represents the total accumulated amount of money spent playing the game n times. Hence a player should pay $m_n - m_{n-1}$, $n \geq 1$ (where $m_0 = 0$), for the n^{th} play of the game. Thus if $E|X| < \infty$ and $EX \neq 0$, we realize that the player should pay the fixed entrance fee EX for each play of the game in order for $\sum_{k=1}^{n} X_k/m_n \to 1$ a.s.

Clearly, by the converse to the Kolmogorov SLLN, if $E|X| = \infty$, then for any fixed entrance fee c, a strong law must fail, i.e.,

$$P\{\lim_{n\to\infty} \frac{\sum_{k=1}^{n} X_k}{cn} = 1\} = 0.$$

Chow and Robbins (1961) improved on this. They established that if $\{X, X_n, n \geq 1\}$ are i.i.d. random variables with $E|X| = \infty$, then

$$P\{\lim_{n\to\infty} \frac{\sum_{k=1}^{n} X_k}{m_n} = 1\} = 0 \tag{55}$$

for all sequences of constants $\{m_n, n \geq 1\}$. This shows that if $E|X| = \infty$, then there is not even a variable entrance fee that makes the game fair in the "strong sense." Equation (55) was first proved by Khintchine (1935) for $\{X_n, n \geq 1\}$ i.i.d. nonnegative random variables whose common distribution function is continuous. By imposing a few minor restrictions on the weights $\{a_n, n \geq 1\}$ we will show that if $E|X| = \infty$, then

$$P\{\lim_{n\to\infty} \frac{S_n}{m_n} = 1\} = 0 \tag{56}$$

for all sequences $\{m_n, n \geq 1\}$. Before proving (56), we need to establish a result that is an extension of Theorem 2 of Feller (1946) to the weighted i.i.d. case.

Theorem 3.5. Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of constants where $0 < b_n \uparrow \infty$, $c_n \uparrow$,

$$c_n/n \to \infty, \tag{57}$$

$$\sum_{k=1}^{n} |a_k| = O(n|a_n|), \tag{58}$$

and

$$\frac{c_n}{n} \leq M \frac{c_j}{j} \quad \text{for some } M > 0 \text{ whenever } j \geq n \geq 1. \tag{59}$$

Then

$$\frac{S_n}{b_n} \to 0 \quad \text{a.s.} \quad \text{iff} \quad \sum_{n=1}^{\infty} P\{|X| > c_n\} < \infty.$$

Proof. The necessity half of this theorem is immediate (see the proof of Theorem 3.4).

We will now prove the sufficiency half. With a little help from Theorem 3.1 we will obtain

$$\frac{\sum_{k=1}^{n} a_k(X_k - EXI(|X| \leq c_k))}{b_n} \to 0 \quad \text{a.s.} \tag{60}$$

as follows. Let $M = D_1 = D_2 = D_3 = 1$. Hence (13) and (14) hold, whence we only need to verify (5) with $q = 2$ in order to establish (15). Using (59) we see that

$$\max_{1 \leq k \leq n} c_k^2 \sum_{j=n}^{\infty} \frac{1}{c_j^2} = c_n^2 \sum_{j=n}^{\infty} \frac{1}{c_j^2} \leq M^2 n^2 \sum_{j=n}^{\infty} \frac{1}{j^2} = M^2 n^2 O(\frac{1}{n}) = O(n).$$

Therefore, via Theorem 3.1, we obtain (15) which by independence is tantamount to (60).

Next, we need to show that

$$\sum_{k=1}^{n} a_k EXI(|X| \leq c_k) = o(b_n). \qquad (61)$$

The conditions (57) and (58) ensure that

$$\frac{\sum_{k=1}^{n} |a_k|}{b_n} \leq \frac{Cn|a_n|}{b_n} = \frac{Cn}{c_n} = o(1). \qquad (62)$$

Also, since $c_n \uparrow$, setting $c_0 = 0$,

$$\sum_{j=1}^{\infty} jP\{c_{j-1} < |X| \leq c_j\} = \sum_{j=1}^{\infty} \sum_{k=1}^{j} P\{c_{j-1} < |X| \leq c_j\}$$

$$= \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} P\{c_{j-1} < |X| \leq c_j\} = \sum_{k=1}^{\infty} P\{|X| > c_{k-1}\}$$

$$\leq 1 + \sum_{k=1}^{\infty} P\{|X| > c_k\} < \infty \qquad (63)$$

by hypothesis. Utilizing (59) and $c_n = b_n/|a_n|$, we obtain $\frac{c_j}{j} \leq \frac{Mb_n}{n|a_n|}$ whenever $n \geq j \geq 1$ or, equivalently,

$$\frac{1}{b_n} \leq \frac{Mj}{n|a_n|c_j} \qquad \text{for } n \geq j \geq 1. \qquad (64)$$

Then for all choices of $n \geq N \geq 1$,

$$\frac{1}{b_n}\left|\sum_{k=1}^{n} a_k EXI(|X| \leq c_k)\right|$$

$$\leq \frac{1}{b_n}\sum_{k=1}^{n} |a_k| E|X| I(|X| \leq c_k)$$

$$\leq \frac{1}{b_n}\sum_{k=1}^{n} |a_k| E|X| I(|X| \leq c_n) \qquad \text{(since } c_n \uparrow\text{)}$$

$$= \frac{1}{b_n}\sum_{k=1}^{n} |a_k| E|X| I(|X| \leq c_N) + \frac{1}{b_n}\sum_{k=1}^{n} |a_k| E|X| I(c_N < |X| \leq c_n)$$

$$\leq \frac{c_N}{b_n}\sum_{k=1}^{n} |a_k| + \sum_{k=1}^{n} |a_k| \sum_{j=N+1}^{n} \frac{1}{b_n} E|X| I(c_{j-1} < |X| \leq c_j)$$

$$= \frac{c_N}{b_n}\sum_{k=1}^{n} |a_k| + \sum_{k=1}^{n} |a_k| \sum_{j=N+1}^{n} \left(\frac{Mj}{n|a_n|c_j}\right) E|X| I(c_{j-1} < |X| \leq c_j) \qquad \text{(by (64))}$$

$$\leq \frac{c_N}{b_n}\sum_{k=1}^{n} |a_k| + M\frac{\sum_{k=1}^{n} a_k}{n|a_n|}\sum_{j=N+1}^{n} jP\{c_{j-1} < |X| \leq c_j\}$$

$$\leq \frac{c_N}{b_n}\sum_{k=1}^{n} |a_k| + C\sum_{j=N+1}^{n} jP\{c_{j-1} < |X| \leq c_j\} \qquad \text{(by (58))}$$

$$\leq \frac{c_N}{b_n} \sum_{k=1}^{n} |a_k| + C \sum_{j=N}^{\infty} jP\{c_{j-1} < |X| \leq c_j\}.$$

First letting $n \to \infty$ shows, via (62), that the first term is $o(1)$. Then letting $N \to \infty$ shows that the second term is $o(1)$ since it is the tail series of a convergent series, recalling (63).

Hence (61) prevails which when combined with (60) yields the desired result. □

Remark. The motivation behind Theorem 3.5 came from Theorem 2 of Feller (1946). Consequently, we may now obtain Feller's Theorem 2 as an immediate corollary but this has already been done (see Corollary 3.1(iii)).

This next corollary will play an integral role in proving Theorem 3.7.

Corollary 3.9. Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables and let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be constants satisfying $0 < b_n \uparrow \infty$, $c_n/n \uparrow \infty$, and (58). Then

$$\frac{S_n}{b_n} \to 0 \quad \text{a.s.} \quad \text{iff} \quad \sum_{n=1}^{\infty} P\{|X| > c_n\} < \infty.$$

Proof. Since $c_n/n \uparrow \infty$, the conditions $c_n \uparrow$, (57), and (59) all obtain. The corollary follows immediately from Theorem 3.5. □

Remark. It is clear that if $|a_n| \uparrow$, then (58) is automatic.

This next corollary is quite similar to Corollary 3.8. There is, however, a trade-off of hypotheses. We no longer assume $EX = 0$, but instead replace (51) by the stronger condition $c_n/n \uparrow \infty$.

Corollary 3.10. Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be constants such that $0 < b_n \uparrow \infty$, $|a_n| \uparrow$, and $c_n/n \uparrow \infty$. Then

$$\frac{S_n}{b_n} \to 0 \quad \text{a.s.} \quad \text{iff} \quad \sum_{n=1}^{\infty} P\{|X| > c_n\} < \infty.$$

Proof. The corollary follows immediately from Corollary 3.9 and the prior remark. □

To accomplish our main goal of this section, namely Theorem 3.7 and (56), we need a few preliminary lemmas. The first one, Lemma 3.7, is in Chow and Robbins (1961), but a simple proof is given here.

Lemma 3.7. Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables. Suppose $0 < c_n/n^{\alpha} \uparrow$ for some $\alpha > 0$.

(i) If $\sum_{n=1}^{\infty} P\{|X| > c_n\} < \infty$, then $\lim_{n\to\infty} \frac{X_n}{c_n} = 0$ a.s.

(ii) If $\sum_{n=1}^{\infty} P\{|X| > c_n\} = \infty$, then $\limsup_{n\to\infty} \frac{|X_n|}{c_n} = \infty$ a.s.

Proof. (i) Assume $\sum_{n=1}^{\infty} P\{|X| > c_n\} < \infty$. Thus by Lemma 3.6 we obtain $\sum_{n=1}^{\infty} P\{|X| > \lambda c_n\} < \infty$ for all $\lambda > 0$, whence by the Borel-Cantelli lemma, $P\{\frac{|X_n|}{c_n} > \lambda \text{ i.o. } (n)\} = 0$ for all $\lambda > 0$. Then, by considering the complementary event,

$$P\{\frac{|X_n|}{c_n} \leq \lambda \text{ eventually } (n)\} = 1 \quad \text{for all } \lambda > 0. \tag{65}$$

Thus

$$P\{\limsup_{n\to\infty} \frac{|X_n|}{c_n} = 0\} = P\{\bigcap_{M=1}^{\infty} [\limsup_{n\to\infty} \frac{|X_n|}{c_n} \leq \frac{1}{M}]\}$$

$$= \lim_{M\to\infty} P\{\limsup_{n\to\infty} \frac{|X_n|}{c_n} \leq \frac{1}{M}\}$$

$$\geq \lim_{M\to\infty} P\{\frac{|X_n|}{c_n} \leq \frac{1}{M} \text{ eventually } (n)\}$$

$$= 1 \qquad\qquad \text{(by (65))}.$$

Hence, we obtain $P\{\limsup_{n\to\infty} \frac{|X_n|}{c_n} = 0\} = 1$, which is tantamount to $P\{\lim_{n\to\infty} \frac{X_n}{c_n} = 0\} = 1$.

(ii) Suppose that $\sum_{n=1}^{\infty} P\{|X| > c_n\} = \infty$. This time we conclude, via Lemma 3.6, that $\sum_{n=1}^{\infty} P\{|X| > \lambda c_n\} = \infty$ for all $\lambda > 0$. Then by the Borel-Cantelli lemma

$$P\left\{\frac{|X_n|}{c_n} > \lambda \text{ i.o. } (n)\right\} = 1 \qquad \text{for all } \lambda > 0. \tag{66}$$

Therefore

$$P\left\{\limsup_{n\to\infty} \frac{|X_n|}{c_n} = \infty\right\} = P\left\{\bigcap_{M=1}^{\infty} \left[\limsup_{n\to\infty} \frac{|X_n|}{c_n} \geq M\right]\right\}$$

$$= \lim_{M\to\infty} P\left\{\limsup_{n\to\infty} \frac{|X_n|}{c_n} \geq M\right\}$$

$$\geq \lim_{M\to\infty} P\left\{\frac{|X_n|}{c_n} > M \text{ i.o. } (n)\right\} = 1 \qquad \text{(by (66))}$$

thereby proving the lemma. □

Lemma 3.8. Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables. Suppose $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ are constants such that $b_n > 0$, $b_n = O(b_{n+1})$, and $c_n/n^{\alpha} \uparrow$ for some $\alpha > 0$. If $\sum_{n=1}^{\infty} P\{|X| > c_n\} = \infty$, then $\limsup_{n\to\infty} \frac{|S_n|}{b_n} = \infty$ a.s.

Proof. Using Lemma 3.7(ii), we obtain $\limsup_{n\to\infty} \frac{|X_n|}{c_n} = \infty$ a.s. where $c_n = b_n/|a_n|$, $n \geq 1$. Thus, for $n \geq 2$

$$\frac{|X_n|}{c_n} = \frac{|a_nX_n|}{b_n} = \frac{|\sum_{k=1}^{n} a_kX_k - \sum_{k=1}^{n-1} a_kX_k|}{b_n}$$

$$\leq \frac{|\sum_{k=1}^{n} a_kX_k|}{b_n} + \frac{b_{n-1}}{b_n} \cdot \frac{|\sum_{k=1}^{n-1} a_kX_k|}{b_{n-1}},$$

whence

$$\limsup_{n\to\infty} \frac{|X_n|}{c_n} \leq \limsup_{n\to\infty} \frac{|\sum_{k=1}^{n} a_kX_k|}{b_n}$$

$$+ \limsup_{n\to\infty}\left(\frac{b_{n-1}}{b_n}\right) \cdot \limsup_{n\to\infty} \frac{|\sum_{k=1}^{n-1} a_kX_k|}{b_{n-1}}$$

$$= \left(\limsup_{n\to\infty} \frac{|S_n|}{b_n}\right)\left(1+\limsup_{n\to\infty}\left(\frac{b_{n-1}}{b_n}\right)\right).$$

Since $\limsup_{n\to\infty} \frac{|X_n|}{c_n} = \infty$ a.s. and $b_{n-1} = O(b_n)$, the result follows. $\square$

Combining these last two lemmas and Corollary 3.9 yields the following lemma.

Lemma 3.9. Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be constants with $0 < b_n \uparrow \infty$ and $c_n/n \uparrow \infty$.

(i) If $\sum_{k=1}^{n} |a_k| = O(n|a_n|)$ and $\sum_{n=1}^{\infty} P\{|X| > c_n\} < \infty$, then

$$\lim_{n\to\infty} \frac{X_n}{c_n} = \lim_{n\to\infty} \frac{S_n}{b_n} = 0 \quad \text{a.s.}$$

(ii) If $\sum_{n=1}^{\infty} P\{|X| > c_n\} = \infty$, then

$$\limsup_{n\to\infty} \frac{|X_n|}{c_n} = \limsup_{n\to\infty} \frac{|S_n|}{b_n} = \infty \quad \text{a.s.}$$

Proof. (i) Lemma 3.7(i) shows that $\lim_{n\to\infty} \frac{X_n}{c_n} = 0$ a.s., while on the other hand Corollary 3.9 demonstrates that $\lim_{n\to\infty} \frac{S_n}{b_n} = 0$ a.s.

(ii) This half of the lemma follows from Lemma 3.7(ii) and Lemma 3.8 since $b_n \uparrow$ implies that $b_n = O(b_{n+1})$. □

Lemma 3.9 allows us to prove the following theorem.

Theorem 3.6. Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables. If $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ are constants such that $0 < b_n \uparrow \infty$, $\sum_{k=1}^{n} |a_k| = O(n|a_n|)$, and $c_n/n \uparrow$, then

$$P\{0 < \limsup_{n\to\infty} \frac{|S_n|}{b_n} < \infty\} = 0.$$

Moreover, if $P\{\limsup_{n\to\infty} \frac{|X_n|}{c_n} < \infty\} > 0$, then

$$\lim_{n\to\infty} \frac{S_n}{b_n} = 0 \quad \text{a.s.}$$

Proof. Note that if $\sum_{n=1}^{\infty} P\{|X| > c_n\} = \infty$, then via Lemma 3.9(ii)

$$P\{0 < \limsup_{n\to\infty} \frac{|S_n|}{b_n} < \infty\}$$

$$\leq P\{\limsup_{n\to\infty} \frac{|S_n|}{b_n} < \infty\} = 0.$$

On the other hand if $\sum_{n=1}^{\infty} P\{|X| > c_n\} < \infty$, then via Lemma 3.9(i), $\lim_{n\to\infty} \frac{S_n}{b_n} = 0$ a.s. Therefore,

$$P\{0 < \limsup_{n\to\infty} \frac{|S_n|}{b_n} < \infty\} = 0$$

and the first part is proved.

Moreover, if $P\{\limsup_{n\to\infty} \frac{|X_n|}{c_n} < \infty\} > 0$ or, equivalently, $P\{\limsup_{n\to\infty} \frac{|X_n|}{c_n} = \infty\} < 1$, then by the contrapositive statement of Lemma 3.9(ii) we obtain $\sum_{n=1}^{\infty} P\{|X| > c_n\} < \infty$. Also since $\sum_{k=1}^{n} |a_k| = O(n|a_n|)$, Lemma 3.9(i) ensures that $\lim_{n\to\infty} \frac{S_n}{b_n} = 0$ a.s. □

We are now ready to prove our principal result of this section. It generalizes some work of Chow and Robbins (1961) and, as will become apparent, its proof owes much to their work.

Theorem 3.7. Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables and $\{a_n, n \geq 1\}$ a sequence of constants. If $E|X| = \infty$, $\sum_{k=1}^{n} |a_k| = O(n|a_n|)$, and $n|a_n| \uparrow$ then for each sequence of constants $\{m_n, n \geq 1\}$ either

$$\liminf_{n\to\infty} \left|\frac{S_n}{m_n}\right| = 0 \text{ a.s.} \quad \text{or} \quad \limsup_{n\to\infty} \left|\frac{S_n}{m_n}\right| = \infty \text{ a.s.}$$

Proof. We can trivially rule out the case when $m_n = 0$ for infinitely many n. Assume that the conclusion fails. Then there is some sequence $\{m_n, n \geq 1\}$ such that

$$P\{\liminf_{n\to\infty} \left|\frac{S_n}{m_n}\right| = 0\} < 1 \quad \text{and} \quad P\{\limsup_{n\to\infty} \left|\frac{S_n}{m_n}\right| = \infty\} < 1 \tag{67}$$

or, equivalently,

$$P\{\liminf_{n\to\infty} \left|\frac{S_n}{m_n}\right| > 0\} > 0 \quad \text{and} \quad P\{\limsup_{n\to\infty} \left|\frac{S_n}{m_n}\right| < \infty\} > 0.$$

Firstly, suppose that $m_n = O(n|a_n|)$. Then since $P\{\limsup_{n\to\infty} \left|\frac{S_n}{m_n}\right| < \infty\} > 0$, we must have $P\{\limsup_{n\to\infty} \left|\frac{S_n}{na_n}\right| < \infty\} > 0$. Therefore $P\{\limsup_{n\to\infty} \left|\frac{S_n}{na_n}\right| = \infty\} < 1$.

Next, we verify the hypotheses of Lemma 3.8 with $b_n = n|a_n|$. So $c_n = b_n/|a_n| = n$. Hence $c_n/n^\alpha \uparrow$ for some $\alpha > 0$ is satisfied and $b_n = O(b_{n+1})$ since $b_n = n|a_n| \uparrow$. Utilizing the contrapositive of Lemma 3.8 we may conclude that $\sum_{n=1}^{\infty} P\{|X| > c_n\} < \infty$. Since $c_n = n$, we have shown that $\sum_{n=1}^{\infty} P\{|X| > n\} < \infty$ which, via (11), is tantamount to $E|X| < \infty$ and this contradicts the hypothesis $E|X| = \infty$.

Alternatively, suppose that

$$\limsup_{n\to\infty} \left|\frac{m_n}{na_n}\right| = \infty. \tag{68}$$

Let $d_n = n \max_{1\le k\le n} \left|\frac{m_k}{ka_k}\right|$, $n \ge 1$. Note that

$$d_n \ge n\left|\frac{m_n}{na_n}\right| = \left|\frac{m_n}{a_n}\right|. \tag{69}$$

Since $\frac{d_n}{n} = \max_{1\le k\le n} \left|\frac{m_k}{ka_k}\right| \ge \left|\frac{m_n}{na_n}\right|$ and (69) holds, we can conclude that

$$0 < \frac{d_n}{n} \uparrow \infty. \tag{70}$$

It will now be shown that $\max_{1\le k\le n} \left|\frac{m_k}{ka_k}\right| = \left|\frac{m_n}{na_n}\right|$ for infinitely many n. Suppose, to the contrary, that the last term is the maximum for only finitely many n. Then there is an integer N such that for every $n \ge N$ there is some $k_n < n$ where

$$\left|\frac{m_{k_n}}{k_n a_{k_n}}\right| = \max_{1 \leq k \leq n} \left|\frac{m_k}{ka_k}\right| > \left|\frac{m_n}{na_n}\right|.$$

Note that

$$\left|\frac{m_{N+1}}{(N+1)a_{N+1}}\right| < \max_{1 \leq k \leq N+1} \left|\frac{m_k}{ka_k}\right| = \max_{1 \leq k \leq N} \left|\frac{m_k}{ka_k}\right|$$

$$= \left|\frac{m_{k_N}}{k_N a_{k_N}}\right|.$$

Likewise

$$\left|\frac{m_{N+2}}{(N+2)a_{N+2}}\right| < \max_{1 \leq k \leq N+2} \left|\frac{m_k}{ka_k}\right| = \max_{1 \leq k \leq N+1} \left|\frac{m_k}{ka_k}\right|$$

$$= \max_{1 \leq k \leq N} \left|\frac{m_k}{ka_k}\right| = \left|\frac{m_{k_N}}{k_N a_{k_N}}\right|.$$

Continuing this argument we see that for every $n \geq N$

$$\frac{d_n}{n} = \max_{1 \leq k \leq n} \left|\frac{m_k}{ka_k}\right| = \left|\frac{m_{k_N}}{k_N a_{k_N}}\right| < \infty$$

thereby contradicting (70). Thus, $\max_{1 \leq k \leq n} \left|\frac{m_k}{ka_k}\right| = \left|\frac{m_n}{na_n}\right|$ for infinitely many n.

Consequently, there is a subsequence $\{d_{n_j}, j \geq 1\}$ with

$$d_{n_j} = n_j \left|\frac{m_{n_j}}{n_j a_{n_j}}\right| = \left|\frac{m_{n_j}}{a_{n_j}}\right|, \qquad j \geq 1.$$

Note, via (69), that $d_n|a_n| \geq |m_n|$ and hence

$$P\{\limsup_{n\to\infty} \left|\frac{S_n}{d_n a_n}\right| < \infty\} \geq P\{\limsup_{n\to\infty} \left|\frac{S_n}{m_n}\right| < \infty\} > 0.$$

Next, we will apply both parts of Lemma 3.9 to obtain $\lim_{n\to\infty} \frac{S_n}{d_n|a_n|} = 0$ a.s. as follows. The norming constants are now $b_n \equiv d_n|a_n|$, $n \geq 1$, whence $c_n = b_n/|a_n| = d_n$, $n \geq 1$. Then $0 < c_n/n \uparrow \infty$, recalling (70). Also we need to verify that $0 < b_n \uparrow \infty$. Now

$$b_n = d_n|a_n| = n|a_n| \max_{1\leq k\leq n} \left|\frac{m_k}{ka_k}\right|,$$

and so $b_n \uparrow$ since $n|a_n| \uparrow$ by hypothesis. To show that $b_n \to \infty$, recalling (68) and again using the fact that $n|a_n| \uparrow$, we see that $\limsup_{n\to\infty} |m_n| = \infty$. Now

$$b_n = n|a_n| \max_{1\leq k\leq n} \left|\frac{m_k}{ka_k}\right| \geq n|a_n| \left|\frac{m_n}{na_n}\right| = |m_n|,$$

and so $\lim_{n\to\infty} b_n \geq \limsup_{n\to\infty} |m_n| = \infty$.

Since we have $P\{\limsup_{n\to\infty} |\frac{S_n}{d_n a_n}| < \infty\} > 0$, we may conclude using the contrapositive of Lemma 3.9(ii) that

$$\sum_{n=1}^{\infty} P\{|X| > d_n\} = \sum_{n=1}^{\infty} P\{|X| > c_n\} < \infty.$$

Finally, note that since $\sum_{k=1}^{n} |a_k| = O(n|a_n|)$ we obtain, via Lemma 3.9(i), that $\lim_{n\to\infty} \frac{S_n}{d_n|a_n|} = 0$ a.s.

Therefore $\lim_{j\to\infty} \frac{S_{n_j}}{d_{n_j}|a_{n_j}|} = 0$ a.s. Since $d_{n_j}|a_{n_j}| = |m_{n_j}|$, $j \geq 1$, we conclude that $\lim_{j\to\infty} |\frac{S_{n_j}}{m_{n_j}}| = 0$ a.s. Thus $\liminf_{n\to\infty} |\frac{S_n}{m_n}| = 0$ a.s. contradicting (67) and thereby proving the theorem under case (68).☐

Using Theorem 3.7 we can now, under the appropriate assumptions, conclude (56).

Corollary 3.11. Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables with $E|X| = \infty$. If $\{a_n, n \geq 1\}$ is a sequence of constants where $\sum_{k=1}^{n} |a_k| = O(n|a_n|)$ and $n|a_n| \uparrow$, then $P\{\lim_{n\to\infty} \frac{S_n}{m_n} = c\} = 0$ for every sequence of constants $\{m_n, n \geq 1\}$ and every finite nonzero constant c.

Proof. Since, by Theorem 3.7 either $\liminf_{n\to\infty} |\frac{S_n}{m_n}| = 0$ a.s. or $\limsup_{n\to\infty} |\frac{S_n}{m_n}| = \infty$ a.s., there is zero probability that $\lim_{n\to\infty} \frac{S_n}{m_n} = c$. ☐

Finally, we note that Theorem 3.7 and Corollary 3.11 generalize the results of Chow and Robbins (1961).

Corollary 3.12 (Chow and Robbins, 1961). If $\{X, X_n, n \geq 1\}$ are i.i.d. random variables with $E|X| = \infty$, then for every sequence of constants $\{m_n, n \geq 1\}$ either

$$\liminf_{n\to\infty} \left|\frac{\sum_{k=1}^{n} X_k}{m_n}\right| = 0 \text{ a.s.} \quad \text{or} \quad \limsup_{n\to\infty} \left|\frac{\sum_{k=1}^{n} X_k}{m_n}\right| = \infty \text{ a.s.}$$

Proof. In view of Theorem 3.7, we need only show that $\sum_{k=1}^{n} |a_k| = O(n|a_n|)$ and $n|a_n| \uparrow$ hold when $a_n \equiv 1$. But these conditions trivially hold, whence the conclusion follows. □

Corollary 3.13 (Chow and Robbins, 1961). If $\{X, X_n, n \geq 1\}$ are i.i.d. random variables with $E|X| = \infty$ then $P\{\lim_{n\to\infty} \frac{\sum_{k=1}^{n} X_k}{m_n} = c\} = 0$ for all sequences of constants $\{m_n, n \geq 1\}$ and every finite nonzero constant c.

Proof. This follows immediately from Corollary 3.12 or Corollary 3.11. □

3.6 Examples

In this section we give a few examples that demonstrate various results of this chapter. Moreover, we present an example that compares a GSLLN with a GCLT.

The following example, which illustrates Theorems 3.1 and 3.2 and Corollary 3.2, has less restrictive conditions than those of Corollary 4 of Teicher (1985).

Example 3.3. Let $\{X, X_n, n \geq 1\}$ be identically distributed random variables and let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be constants with $0 < b_n \uparrow \infty$. Suppose that $b_n/|a_n| = (nL(n))^{1/p}$, $n \geq 1$, for some $0 < p < 2$ where L is a positive slowly varying function such that either

(i) $L((xL(x))^{1/p}) = O(L(x))$ as $x \to \infty$, $x^p/L(x)$ is eventually nondecreasing, and $E\frac{|X|^p}{L(|X|)} < \infty$

or

(ii) $\liminf\limits_{x\to\infty} L(x) > 0$ and $E|X|^p < \infty$.

Moreover, suppose that if $1 \leq p < 2$ then $\{X, X_n, n \geq 1\}$ are independent with $EX = 0$. Furthermore, suppose (28) holds if $p = 1$ and $xL(x)$ is eventually nondecreasing if $1 < p < 2$. Then

$$\frac{\sum\limits_{k=1}^{n} a_k X_k}{b_n} \to 0 \quad \text{a.s.} \tag{71}$$

Proof. Let $c_n = (nL(n))^{1/p}$, $n \geq 1$. We will firstly show that

$$\sum_{n=1}^{\infty} P\{|X| > c_n\} < \infty.$$

Under (i), there exists an $\varepsilon > 0$ and a positive integer N such that for all $n \geq N$

$$\frac{L(n)}{L((nL(n))^{1/p})} > \varepsilon \tag{72}$$

and

$$x^p/L(x) \uparrow \qquad \text{for } x \geq N. \tag{73}$$

Therefore,

$$\sum_{n=1}^{\infty} P\{|X| > c_n\}$$

$$= C + \sum_{n=N}^{\infty} P\{|X| > (nL(n))^{1/p}\}$$

$$\leq C + \sum_{n=N}^{\infty} P\{\frac{|X|^p}{L(|X|)} \geq \frac{nL(n)}{L((nL(n))^{1/p})}\} \qquad \text{(by (73))}$$

$$\leq C + \sum_{n=N}^{\infty} P\{\frac{|X|^p}{L(|X|)} > \varepsilon n\} \qquad \text{(by (72))}$$

$$< \infty \qquad \text{(by (11))}.$$

On the other hand, under (ii), there exists an $\varepsilon > 0$ and a positive integer N such that for all $n \geq N$

$$L(n) \geq \varepsilon \tag{74}$$

Hence,

$$\sum_{n=1}^{\infty} P\{|X| > c_n\}$$

$$= C + \sum_{n=N}^{\infty} P\{|X| > (nL(n))^{1/p}\}$$

$$\leq C + \sum_{n=N}^{\infty} P\{|X|^p > \varepsilon n\} \qquad \text{(by (74))}$$

$$< \infty \qquad \text{(by (11))}.$$

Observe that via slow variation of L

$$\liminf_{n\to\infty} \frac{c_{2n}}{c_n} = \liminf_{n\to\infty} \frac{(2nL(2n))^{1/p}}{(nL(n))^{1/p}}$$

$$= 2^{1/p} > 2$$

whenever $0 < p < 1$. So if $0 < p < 1$, then invoking the sufficiency part of Lemma 3.5 with q = 1 and r = 2, the hypotheses of Theorem 3.2(i) are satisfied and so (71) follows.

Next, observe that

$$\liminf_{n\to\infty} \frac{c_{2n}^2}{c_n^2} = \liminf_{n\to\infty} \frac{(2nL(2n))^{2/p}}{(nL(n))^{2/p}}$$

$$= 2^{2/p} > 2$$

whenever $0 < p < 2$. Then for any $0 < p < 2$, (5) holds with $q = 2$ via Lemma 3.5 with $r = 2$. So if $p = 1$, (71) follows by the sufficiency half of Corollary 3.2.

Finally, suppose that $1 < p < 2$. Note that since $xL(x)$ is eventually nondecreasing, we obtain (16). We will now verify (17). Since $xL(x)$ is eventually nondecreasing, there exists a positive integer N such that if $n \geq N$ then

$$nL(n) \uparrow. \tag{75}$$

Thus, for all $n \geq N+1$,

$$c_n \sum_{k=1}^{n} \frac{1}{c_k} = (nL(n))^{1/p} \sum_{k=1}^{n} \frac{1}{(kL(k))^{1/p}}$$

$$= (nL(n))^{1/p} \sum_{k=1}^{N} \frac{1}{(kL(k))^{1/p}} + (nL(n))^{1/p} \sum_{k=N+1}^{n} \frac{1}{(kL(k))^{1/p}}$$

$$\leq C(nL(n))^{1/p} + (nL(n))^{1/p}\int_1^n \frac{1}{(xL(x))^{1/p}}dx \qquad \text{(by (75))}$$

$$\leq C(nL(n))^{1/p} + (1+o(1))\left(\frac{p}{p-1}\right)\left(\frac{nL(n)}{L(n)}\right)^{1/p} n^{\frac{p-1}{p}} \qquad \text{(by Theorem 1(b) of Feller, 1971, p. 281)}$$

$$= O(n) \qquad \text{(recall } 1 < p < 2 \text{ and } L(x) = o(x^{\alpha}) \text{ for all } \alpha > 0\text{).}$$

Thus, (17) holds and so (18) obtains by Theorem 3.1. Then (71) follows by independence and $EX_n = EX = 0$, $n \geq 1$. □

Remarks. Note that the function $L(x) = (\log x)^{\alpha}$ (where $-\infty < \alpha < \infty$) clearly satisfies the first two conditions of (i). Thus we have generalized Corollary 4 of Teicher (1985) wherein $a_n = n^{-1/p}$, $n \geq 1$, and $b_n = (\log n)^{1/p}$, $n \geq 1$, produce $c_n = (n\log n)^{1/p}$, $n \geq 1$, in his first part (i.e., $\alpha = 1$) and $a_n = n^{\alpha}$, $n \geq 1$, and $b_n = n^{\alpha+(1/p)}$, $n \geq 1$, generate $c_n = n^{1/p}$, $n \geq 1$, in his second part (i.e., $\alpha = 0$).

It should also be noted that the condition $L((xL(x))^{1/p}) = O(L(x))$ as $x \to \infty$ for some $0 < p < 2$ is not automatic for every slowly varying function L. For Rosalsky (1987) showed that $L(xL(x))/L(x) \to \infty$ as $x \to \infty$ for the function $L(x) = \exp\{(\log_2 x)^{-1}\log x\}$.

This next example will compare results from this chapter and the previous one by reexamining Example 2.2. Notice the relative freedom we have in choosing the sequence $\{c_n, n \geq 1\}$.

Example 3.4. Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables with common density function

$$f(x) = \begin{cases} \frac{e^2}{2}\left(\frac{2(\log|x|)-1}{|x|^3}\right) & \text{for } |x| \geq e, \\ 0 & \text{for } |x| < e. \end{cases}$$

Then $\sum_{k=1}^{n} a_k X_k / b_n \to 0$ a.s. for every choice of sequences $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ provided that $0 < b_n \uparrow \infty$ and $c_n = n^{1/p} L(n) \uparrow$ for some $0 < p < 2$ and positive, slowly varying function L.

Proof. We have already shown in Example 2.2 that $EX = 0$ and $P\{|X| > s\}$ is regularly varying with exponent -2. So recalling Theorem 3.4, we need only show that (5) holds with $q = 2$ and that $\sum_{n=1}^{\infty} P\{|X| > c_n\} < \infty$. It is clear that $X \in \mathcal{L}_r$ for all $0 < r < 2$. Let $0 < \varepsilon < \frac{2-p}{2p}$ and, since L is slowly varying, $(L(n))^{-1} \leq n^{\varepsilon}$ for all $n \geq$ some n_o. Therefore

$$\sum_{n=1}^{\infty} P\{|X| > c_n\} = \sum_{n=1}^{\infty} P\{|X| > n^{1/p} L(n)\}$$

$$\leq C + \sum_{n=n_o}^{\infty} P\{|X| > n^{\frac{1-\varepsilon p}{p}}\} < \infty \qquad \text{(by choice of } \varepsilon \text{ and (11)).}$$

Next, observe that

$$\max_{1\le k\le n} c_k^2 \sum_{j=n}^{\infty} \frac{1}{c_j^2} = n^{2/p}(L(n))^2 \sum_{j=n}^{\infty} \frac{1}{j^{2/p}(L(j))^2}$$

$$\le n^{2/p}(L(n))^2 \int_{n-1}^{\infty} \frac{1}{x^{2/p}(L(x))^2} dx$$

$$= (1+o(1))n^{2/p} \frac{(L(n))^2 p}{(L(n-1))^2(2-p)(n-1)^{(2-p)/p}}$$

(by Theorem 1(a) of Feller (1971), p. 281)

$$= O(n) \qquad \text{(by Lemma 2.9)}$$

thereby establishing (5) with $q = 2$. □

Remark. As must be the case, the sequences in Example 2.2 must fail to satisfy the hypotheses in Example 3.4. Recall in Example 2.2 that $a_n = L(n)$, $n \ge 1$, where $L(x)$ is any positive slowly varying, nondecreasing function and $b_n = \frac{e}{2}\sqrt{n}L(n)\log n$, $n \ge 1$. Hence

$$c_n = b_n/|a_n| = \frac{\frac{e}{2}\sqrt{n}L(n)\log n}{L(n)} = \frac{e}{2}\sqrt{n}\log n, \; n \ge 1,$$

does not satisfy the condition that $c_n = n^{1/p}L^*(n) \uparrow$ for some $0 < p < 2$ and positive slowly varying function L^*. Moreover, it does not satisfy the hypothesis that (5) holds with $q = 2$ since

$$\max_{1\le k\le n} c_k^2 \sum_{j=n}^{\infty} \frac{1}{c_j^2} = n(\log n)^2 \sum_{j=n}^{\infty} \frac{1}{j(\log j)^2} \sim n\log n \neq O(n).$$

As it has been noted throughout this chapter, there have been many studies generalizing the SLLN. One such investigation by Rohatgi (1971) has a few conditions which are similar to some of ours. For example, Rohatgi assumed the existence of a stochastically dominating random variable, but instead of considering the normalization of partial sums of weighted random variables he studied the sequence $\{\sum_{k=1}^{n} a_{nk}X_k, n \ge 1\}$ which is more general than our $\{\sum_{k=1}^{n} a_kX_k/b_n, n \ge 1\}$. Rohatgi's main result may be stated as follows.

Theorem 3.8 (Rohatgi, 1971). Let $\{X_n, n \ge 1\}$ be a sequence of independent random variables with $E|X_n| < \infty$ and $EX_n = 0$ for $n \ge 1$ and satisfying for some random variable X

$$P\{|X_n| > x\} \le P\{|X| > x\} \quad \text{for all } x > 0,\ n \ge 1. \tag{76}$$

Let $\{a_{nk}, n \ge 1, k \ge 1\}$ be a double array of constants which satisfy

$$\lim_{n\to\infty} a_{nk} = 0 \qquad \text{for all } k \ge 1, \tag{77}$$

and

$$\sum_{k=1}^{\infty} |a_{nk}| = O(1). \tag{78}$$

If

$$\max_{k \geq 1} |a_{nk}| = O(n^{-v}) \quad \text{for some } v > 0, \tag{79}$$

and

$$E|X|^{1+\frac{1}{v}} < \infty \tag{80}$$

then

$$\sum_{k=1}^{\infty} a_{nk} X_k \to 0 \quad \text{a.s.} \tag{81}$$

Proof. See Theorem 2 of Rohatgi (1971). □

It is natural to ask if a result as fundamental as the classical Kolmogorov SLLN follows from Rohatgi's work. To be precise, for $\{X_n, n \geq 1\}$ i.i.d. mean zero random variables, is

$$\sum_{k=1}^{n} X_k/n \to 0 \quad \text{a.s.} \tag{82}$$

a consequence of Theorem 3.8? The Kolmogorov SLLN is established by many of the results of this chapter. For example we can use Corollary 3.1(i) or (ii) to obtain the Kolmogorov SLLN.

The point here is that this basic SLLN is not covered by Theorem 3.8. In this situation

$$a_{nk} = \begin{cases} \frac{1}{n} & \text{if } 1 \leq k \leq n, \\ 0 & \text{if } k > n. \end{cases}$$

Conditions (76), (77), and (78) are all satisfied. The major drawbacks to Theorem 3.8 are the conditions (79) and (80).

Condition (78) states that $1/n = 0(n^{-v})$, which implies that $0 < v \leq 1$. On the other hand, (80) states that $E|X_1|^{1+\frac{1}{v}} < \infty$. This means that for (82) to hold via Theorem 3.8 we need $X_1 \varepsilon \mathcal{L}_p$ for some $p \geq 2$, but this, as we have seen, is far too stringent a requirement.

This example shows the inherent competition between conditions (79) and (80). Condition (79) is weakened when v is small, while condition (80) is weakened when v is large.

The hypotheses we have used usually have reinforced each other. As in Example 3.4, we saw that $c_n = n^{1/p}L(n)$ satisfies both (5) with $q = 2$ and $\sum_{n=1}^{\infty} P\{|X| > c_n\} < \infty$.

A comparison between Theorem 3.8 and Corollary 3.2 deserves some attention. In both cases we have independent random variables with $EX_n = 0$, $n \geq 1$. If we let

$$a_{nk} = \begin{cases} |a_k|/b_n & \text{if } 1 \le k \le n, \\ 0 & \text{if } k > n, \end{cases}$$

we see that (77) must hold provided $b_n \to \infty$. It is clear that (78) is equivalent to (28). Likewise (79) and (80) are similar in spirit to (5) and (30), respectively.

Finally, we will end this section with an example that illustrates the differences between Theorems 3.2 and 3.8. We present a nontrivial example, since it is clear that (80) fails if we assume $E|X|^r = \infty$ for some $r \le 1$.

Example 3.5. Let $\{X_n, n \ge 1\}$ and X be random variables such that $\{|X_n|, n \ge 1\}$ is stochastically dominated by $|X|$ in the sense that there exists a constant $D < \infty$ satisfying (32). Suppose that $E|X|^{3/2} < \infty$ but $E|X|^r = \infty$ for $r > 3/2$. If $\{a_n, n \ge 1\}$ and $\{b_n, n \ge 1\}$ are constants with $0 < b_n \uparrow \infty$ and $c_n = n^{3/2}$, $n \ge 1$, then $\sum_{k=1}^{n} a_k X_k / b_n \to 0$ a.s.

Proof. In view of Theorem 3.2 we only need to verify (5) with $q = 1$ and (33). Condition (5) holds since

$$\max_{1 \le k \le n} c_k \sum_{j=n}^{\infty} \frac{1}{c_j} = n^{3/2} \sum_{j=n}^{\infty} \frac{1}{j^{3/2}} = n^{3/2} O(n^{-1/2}) = O(n).$$

Next, observe that

$$\sum_{n=1}^{\infty} P\{|X| > Dc_n\} = \sum_{n=1}^{\infty} P\{\frac{|X|}{D} > n^{3/2}\}$$

$$\leq \sum_{n=1}^{\infty} P\{\frac{|X|}{D} > n^{2/3}\} < \infty \qquad \text{(by (11))} \qquad (83)$$

and hence (33) also holds. □

Remark. First, note that the inequality in (83) is very crude. We can easily weaken the moment condition on X and still obtain the desired result. Next, note that we are not assuming that the random variables are independent nor that $EX_n = 0$, $n \geq 1$, as in Theorem 3.8. Also, the version of stochastic domination is more general in Theorem 3.2 than in Theorem 3.8.

Finally, note that we have a lot of freedom in choosing the weights $\{a_n, n \geq 1\}$ and the normalizing constants $\{b_n, n \geq 1\}$. For example, suppose $a_n = \frac{1}{\sqrt{n}}$, $n \geq 1$, and $b_n = n$, $n \geq 1$. This clearly satisfies $0 < b_n \uparrow \infty$ and $c_n = n^{3/2}$, but we will now see that Theorem 3.8 fails to establish the GSLLN. In this situation, let

$$a_{nk} = \begin{cases} \frac{1}{\sqrt{kn}} & \text{if } 1 \leq k \leq n, \\ 0 & \text{if } k > n. \end{cases} \qquad (84)$$

Fact. Define $\{a_{nk}, k \geq 1, n \geq 1\}$ as in (84). If $E|X|^{3/2} < \infty$ but $E|X|^r = \infty$ for $r > 3/2$, then (79) and (80) cannot hold simultaneously (in this example).

Proof. Condition (79) reduces to $\frac{1}{n} = 0(n^{-v})$ for some $v > 0$ which implies that $v \leq 1$. Next, if (80) is to hold, then, in view of our moment requirement on X, we have $1 + \frac{1}{v} \leq \frac{3}{2}$ or equivalently $v \geq 2$. Thus the conditions (79) and (80) cannot both hold. □

This example shows the intrinsic conflict between the conditions (79) and (80).

CHAPTER FOUR
A GENERALIZED WEAK LAW OF LARGE NUMBERS

4.1 Introduction

The purpose of this chapter is to propose conditions where a Generalized Weak Law of Large Numbers (GWLLN) obtains, i.e.,

$$\frac{\sum_{k=1}^{n} a_k X_k}{b_n} \xrightarrow{P} 0. \tag{1}$$

On one hand our job in some sense is quite easy, while on the other hand it is very difficult. The easy part comes from the fact that almost sure convergence implies convergence in probability. Therefore all the GSLLN results in Chapter Three are also GWLLN results. So it will be our goal to change some of the assumptions placed on the random variables and constants. What makes our work difficult is that the weak law has been developed in great depth, see Chow and Teicher (1978, p. 326-329), or Loève (1977, p. 329). To this end, we will strengthen the hypotheses placed on the random variables $\{X, X_n, n \geq 1\}$ while weakening the conditions imposed on the constants $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$. As in Chapter Three, set $c_n = b_n/|a_n|, n \geq 1$.

4.2 Mainstream

Our major result in this chapter, Theorem 4.1, has two motivating factors. The first part of Theorem 4.1 is an adaptation of a result of Klass and Teicher (1977) to the case where the weight sequence $\{a_n, n \geq 1\}$ is not constant. Meanwhile, the other half of Theorem 4.1 is a refinement of Theorem 3 from Chow and Teicher (1978, p. 123-124).

Theorem 4.1. Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be constants such that $a_n \neq 0$, $b_n > 0$, $n \geq 1$.

(i) Suppose that

$$\sum_{k=1}^{n} a_k^2 = o(b_n^2) \tag{2}$$

and either

$$(\alpha) \quad c_n \uparrow, \; c_n/n \downarrow, \text{ and } \sum_{k=1}^{n} (c_k/k)^2 = O\left(\frac{b_n^2}{\sum_{k=1}^{n} a_k^2}\right) \tag{3}$$

or

$$(\beta) \quad c_n/n \uparrow \text{ and } \sum_{k=1}^{n} a_k^2 = O(na_n^2). \tag{4}$$

Then if

$$nP\{|X| > c_n\} = o(1) \tag{5}$$

the GWLLN

$$\frac{\sum_{k=1}^{n} a_k(X_k - EXI(|X| \le c_n))}{b_n} \xrightarrow{P} 0 \tag{6}$$

obtains.

(ii) Conversely, if $\{X, X_n, n \ge 1\}$ are independent random variables, and $\{a_n, n \ge 1\}$ and $\{b_n, n \ge 1\}$ are constants with $0 < b_n \uparrow \infty$, $c_n \to \infty$ and

$$\inf\{|\frac{a_k}{a_n}| : 1 \le k \le n, n \ge 1\} > 0, \tag{7}$$

then (6) implies (5).

Proof. (i) Let $d_n = c_n/n$, $n \ge 1$ and $X_k^{(n)} = X_k I(|X_k| \le c_n)$ for $1 \le k \le n$, $n \ge 1$. Also let $S_n = \sum_{k=1}^{n} a_k X_k$, $n \ge 1$, and $S_n' = \sum_{k=1}^{n} a_k X_k^{(n)}$, $n \ge 1$. Then for arbitrary $\varepsilon > 0$

$$P\{\frac{|S_n - ES_n'|}{b_n} > \varepsilon\} = P\{\frac{|S_n' - ES_n' + S_n - S_n'|}{b_n} > \varepsilon\}$$

$$\le P\{\frac{|S_n' - ES_n'|}{b_n} > \frac{\varepsilon}{2}\} + P\{\frac{|S_n - S_n'|}{b_n} > \frac{\varepsilon}{2}\}. \tag{8}$$

Now

$$P\{\frac{|S_n - S_n'|}{b_n} > \frac{\varepsilon}{2}\} \le P\{S_n \ne S_n'\} = P\{\bigcup_{k=1}^{n} [X_k^{(n)} \ne X_k]\}$$

$$\leq \sum_{k=1}^{n} P\{|X_k| > c_n\} = nP\{|X| > c_n\} = o(1) \qquad \text{(by (5))}. \qquad (9)$$

Letting $c_0 = 0$ and noting that $c_n \uparrow$ under either case (α) or (β), it follows that

$$P\{\frac{|S'_n - ES'_n|}{b_n} > \frac{\varepsilon}{2}\}$$

$$\leq \frac{4}{\varepsilon^2 b_n^2} \text{Var}(\sum_{k=1}^{n} a_k X_k^{(n)})$$

$$= \frac{C}{b_n^2} \sum_{k=1}^{n} a_k^2 \text{Var}(X_k^{(n)})$$

$$\leq \frac{C}{b_n^2} \sum_{k=1}^{n} a_k^2 EX^2 I(|X| \leq c_n)$$

$$= (\frac{C}{b_n^2} \sum_{k=1}^{n} a_k^2) \sum_{j=1}^{n} EX^2 I(c_{j-1} < |X| \leq c_j)$$

$$\leq (\frac{C}{b_n^2} \sum_{k=1}^{n} a_k^2) \sum_{j=1}^{n} c_j^2 P\{c_{j-1} < |X| \leq c_j\}$$

$$= (\frac{C}{b_n^2} \sum_{k=1}^{n} a_k^2) \sum_{j=1}^{n} c_j^2 [P\{|X| > c_{j-1}\} - P\{|X| > c_j\}]$$

$$= \left(\frac{C}{b_n^2}\sum_{k=1}^{n} a_k^2\right)\left[c_1^2 P\{|X| > 0\} - c_n^2 P\{|X| > c_n\} + \sum_{j=1}^{n-1}(c_{j+1}^2 - c_j^2)P\{|X| > c_j\}\right]$$

(by Abel's summation by parts lemma)

$$\leq \left(\frac{C}{b_n^2}\sum_{k=1}^{n} a_k^2\right)\sum_{j=1}^{n-1}(c_{j+1}^2 - c_j^2)P\{|X| > c_j\} + \frac{C}{b_n^2}\sum_{k=1}^{n} a_k^2$$

$$= \left(\frac{C}{b_n^2}\sum_{k=1}^{n} a_k^2\right)\sum_{j=1}^{n-1}[(j+1)^2 d_{j+1}^2 - j^2 d_j^2]P\{|X| > c_j\} + o(1) \qquad \text{(by (2))}$$

$$\leq \left(\frac{C}{b_n^2}\sum_{k=1}^{n} a_k^2\right)\sum_{j=1}^{n-1}[j^2 d_{j+1}^2 + 3jd_{j+1}^2 - j^2 d_j^2]P\{|X| > c_j\} + o(1)$$

$$= \left(\frac{C}{b_n^2}\sum_{k=1}^{n} a_k^2\right)\sum_{j=1}^{n-1}[j(d_{j+1}^2 - d_j^2) + 3d_{j+1}^2]jP\{|X| > c_j\} + o(1)$$

$$= C\sum_{j=0}^{n} B_{nj}\, jP\{|X| > c_j\} + o(1) \qquad (10)$$

where

$$B_{nj} = \begin{cases} \left(\frac{1}{b_n^2}\sum_{k=1}^{n} a_k^2\right)[j(d_{j+1}^2 - d_j^2) + 3d_{j+1}^2] & \text{if } j=1,\ldots,n-1, \\ 0 & \text{if } j = 0,n. \end{cases}$$

It will now be shown that $\{B_{nj},\ 0 \leq j \leq n,\ n \geq 1\}$ is a Toeplitz array, that is

$$\sum_{j=0}^{n} |B_{nj}| = O(1) \tag{11}$$

and

$$B_{nj} \to 0 \quad \text{as } n \to \infty \text{ for all fixed } j \geq 0. \tag{12}$$

First note that

$$B_{nj} = \left(\frac{\sum_{k=1}^{n} a_k^2}{b_n^2}\right)[j(d_{j+1}^2 - d_j^2) + 3d_{j+1}^2]$$

$$\geq \left(\frac{\sum_{k=1}^{n} a_k^2}{b_n^2}\right)\frac{1}{j}(c_{j+1}^2 - c_j^2) \geq 0 \qquad (\text{since } c_j \uparrow).$$

Clearly (12) prevails by (2). In case (α) we note that $d_n \downarrow$ and $\sum_{j=1}^{n} d_j^2 = O(b_n^2/\sum_{k=1}^{n} a_k^2)$, whence

$$\sum_{j=0}^{n} |B_{nj}| = \sum_{j=0}^{n} B_{nj} \leq \sum_{j=1}^{n-1} \left(\frac{\sum_{k=1}^{n} a_k^2}{b_n^2}\right) 3d_{j+1}^2 \leq 3\left(\frac{\sum_{k=1}^{n} a_k^2}{b_n^2}\right) \sum_{j=1}^{n} d_j^2$$

$$= O(1),$$

whence (11) obtains.

On the other hand, under case (β), we have $d_n \uparrow$ and $\sum_{k=1}^{n} a_k^2 \leq Cna_n^2$, $n \geq 1$. Then for all $n \geq 1$

$$\frac{\sum_{k=1}^{n} a_k^2}{b_n^2} \leq \frac{Cna_n^2}{b_n^2} = \frac{Cn}{c_n^2} = \frac{Cn}{(nd_n)^2} = \frac{C}{nd_n^2}.$$

Then

$$\sum_{j=0}^{n} |B_{nj}| = \sum_{j=0}^{n} B_{nj}$$

$$= \sum_{j=1}^{n-1} \left(\frac{\sum_{k=1}^{n} a_k^2}{b_n^2}\right)[j(d_{j+1}^2 - d_j^2) + 3d_{j+1}^2]$$

$$\leq \frac{C}{nd_n^2} \sum_{j=1}^{n-1} (j(d_{j+1}^2 - d_j^2) + 3d_{j+1}^2)$$

$$= \frac{C}{nd_n^2}\left(\sum_{j=1}^{n-1} jd_{j+1}^2 - \sum_{j=1}^{n-1} jd_j^2\right) + \frac{C}{nd_n^2}\sum_{j=1}^{n-1} d_{j+1}^2$$

$$\leq \frac{C}{nd_n^2}\left(\sum_{j=2}^{n} (j-1)d_j^2 - \sum_{j=2}^{n-1} jd_j^2\right) + \frac{C}{nd_n^2}(n-1)d_n^2 \qquad (\text{since } d_n \uparrow)$$

$$\leq \frac{C}{nd_n^2}\left(\sum_{j=2}^{n} jd_j^2 - \sum_{j=2}^{n-1} jd_j^2\right) + O(1)$$

$$= \frac{C}{nd_n^2}(nd_n^2) + O(1) = O(1)$$

and again (11) holds thereby proving that $\{B_{nj}, 0 \leq j \leq n, n \geq 1\}$ is a Toeplitz array.

Then by (5) and the Toeplitz lemma (see, e.g., Knopp, 1951, p. 74)

$$\sum_{j=0}^{n} B_{nj} jP\{|X| > c_j\} = o(1). \tag{13}$$

Retracing our steps we see that (10) and (13) ensure that $P\{\frac{|S_n'-ES_n'|}{b_n} > \frac{\varepsilon}{2}\} = o(1)$ which together with (8) and (9) imply that $\frac{S_n-ES_n'}{b_n} \xrightarrow{P} 0$ thereby proving (6).

(ii) Define $\delta = \inf\{|\frac{a_k}{a_n}| : 1 \leq k \leq n, n \geq 1\}$. Therefore for all $n \geq 1$ and $1 \leq k \leq n$, $|a_k| \geq \delta|a_n|$ or, equivalently,

$$\frac{\delta}{|a_k|} \leq \frac{1}{|a_n|} \qquad \text{for } 1 \leq k \leq n, n \geq 1. \tag{14}$$

Set

$$D_n = \sum_{k=1}^{n} a_k EXI(|X| \leq c_n), \ n \geq 1,$$

$$S_n = \sum_{k=1}^{n} a_k X_k, \ n \geq 1,$$

$$d_n = D_n - D_{n-1}, \ n \geq 2.$$

Thus, for $n \geq 2$

$$\frac{|a_n X_n - d_n|}{b_n} = \frac{|(S_n - S_{n-1}) - (D_n - D_{n-1})|}{b_n}$$

$$= \left|\frac{S_n - D_n}{b_n} - \frac{b_{n-1}}{b_n} \cdot \frac{S_{n-1} - D_{n-1}}{b_{n-1}}\right|$$

$$\leq \frac{|S_n - D_n|}{b_n} + \frac{|S_{n-1} - D_{n-1}|}{b_{n-1}} \qquad \text{(since } b_n \uparrow\text{)}$$

$$\xrightarrow{P} 0 \qquad \text{(by (6))}.$$

Then since $X_n \overset{d}{=} X$, we have $\frac{a_n X}{b_n} - \frac{d_n}{b_n} \xrightarrow{P} 0$. The hypothesis $c_n \to \infty$ ensures that $\frac{a_n X}{b_n} \xrightarrow{P} 0$ which then implies that $d_n/b_n = o(1)$.

For arbitrary $\varepsilon > 0$, by Lévy's inequality (letting $m(Y)$ denote a median of a random variable Y),

$$P\{\max_{1 \leq k \leq n} |(S_k - D_k) - m(S_k - D_k - S_n + D_n)| > \varepsilon b_n\}$$

$$\leq 2P\{|S_n - D_n| > \varepsilon b_n\} = o(1) \qquad \text{(by (6))}.$$

Hence

$$\frac{\max_{1 \leq k \leq n} |(S_k - D_k) - m(S_k - D_k - S_n + D_n)|}{b_n} \xrightarrow{P} 0. \tag{15}$$

Next, it follows from (6) and Exercise 3.3.7 of Chow and Teicher (1978, p. 73) that

$$\max_{1 \leq k \leq n} |m(S_k - D_k - S_n + D_n)| = o(b_n). \tag{16}$$

Therefore

$$\frac{\max_{1 \leq k \leq n} |S_k - D_k|}{b_n}$$

$$\leq \frac{\max_{1 \leq k \leq n} |(S_k - D_k) - m(S_k - D_k - S_n + D_n)|}{b_n}$$

$$+ \frac{\max_{1 \leq k \leq n} |m(S_k - D_k - S_n + D_n)|}{b_n} \xrightarrow{P} 0 \qquad \text{(by (15) and (16))}. \tag{17}$$

We now show that

$$\max_{1 \le k \le n} |d_k| = o(b_n). \tag{18}$$

Recalling that $d_n = o(b_n)$ and $b_n \uparrow$, we obtain for all $m < n$

$$\frac{\max\limits_{1 \le k \le n} |d_k|}{b_n} \le \frac{\max\limits_{1 \le k \le m} |d_k|}{b_n} + \frac{\max\limits_{m < k \le n} |d_k|}{b_n}$$

$$\le \frac{\max\limits_{1 \le k \le m} |d_k|}{b_n} + \max_{m < k \le n} \left|\frac{d_k}{b_k}\right| \xrightarrow{n \to \infty} \sup_{k > m} \left|\frac{d_k}{b_k}\right| \xrightarrow{m \to \infty} 0$$

thereby proving (18).

Recalling (7), note that $\delta > 0$. Then realizing that

$$P\{\max_{1 \le k \le n} |a_k X_k - d_k| \le \tfrac{2}{3}\delta b_n\}$$

$$\ge P\{\max_{1 \le k \le n} |S_k - D_k| \le \tfrac{\delta}{3} b_n\},$$

we conclude, via (17), that

$$\lim_{n \to \infty} P\{\max_{1 \le k \le n} |a_k X_k - d_k| \le \tfrac{2}{3}\delta b_n\} = 1. \tag{19}$$

Using (18), we have for sufficiently large n

$$P\{\max_{1\leq k\leq n}|a_kX_k-d_k| \leq \tfrac{2}{3}\delta b_n\}$$

$$\leq P\{\max_{1\leq k\leq n}|a_kX_k| \leq \delta b_n\}$$

$$= P\{\bigcap_{k=1}^{n}[|a_kX_k| \leq \delta b_n]\}$$

$$= \prod_{k=1}^{n} P\{|X| \leq \delta b_n/|a_k|\}$$

$$\leq \prod_{k=1}^{n} P\{|X| \leq b_n/|a_n|\} \qquad \text{(by (14))}$$

$$= (P\{|X| \leq c_n\})^n = (1 - P\{|X| > c_n\})^n.$$

Recalling (19) we obtain

$$\lim_{n\to\infty} (1 - P\{|X| > c_n\})^n = 1.$$

Using the basic inequality $1-s \leq e^{-s}$, we see that

$$(1 - P\{|X| > c_n\})^n \leq \exp\{-nP\{|X| > c_n\}\} \leq 1$$

and thus (5) follows. $\square$

The hypotheses of Theorem 4.1 do not seem to be directly comparable to the hypotheses of the strong law theorems. The condition (4) does resemble the previous condition $\sum_{k=1}^{n} |a_k| = O(n|a_n|)$, but there are in fact other similarities.

Fact. If X is any random variable and $\{c_n, n \geq 1\}$ is a nondecreasing sequence of constants satisfying $\sum_{n=1}^{\infty} P\{|X| > c_n\} < \infty$, then (5) follows.

Proof. Since $c_n \uparrow$ we have $P\{|X| > c_n\} \downarrow$. This combined with $\sum_{n=1}^{\infty} P\{|X| > c_n\} < \infty$ implies via Knopp (1951, p. 124) that $nP\{|X| > c_n\} = o(1)$. □

An example (Example 4.2) will be given in Section 4.3 showing that we can obtain via Theorem 4.1 a GWLLN. The example fails to satisfy the hypotheses of the GSLLN of Theorem 3.5. In fact, we show that the GSLLN actually fails. Before we present our examples, a few immediate corollaries of Theorem 4.1 will be established.

Corollary 4.1 (Klass and Teicher, 1977). Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables and let $\{b_n, n \geq 1\}$ be constants such that $0 < b_n \uparrow$ and either

(i) $b_n/n \downarrow$, $b_n/\sqrt{n} \to \infty$, and $\sum_{k=1}^{n} (b_k/k)^2 = O(b_n^2/n)$

or

(ii) $b_n/n \uparrow$.

Then

$$\frac{\sum_{k=1}^{n} X_k - nEXI(|X| \le b_n)}{b_n} \xrightarrow{P} 0 \tag{20}$$

iff

$$nP\{|X| > b_n\} = o(1). \tag{21}$$

Proof. Assume that (21) holds. Now letting $a_n \equiv 1$, we note that in either case (i) or (ii) we have $b_n^2/n \to \infty$ which is tantamount to (2). Finally, note that (3) trivially holds under case (i) and that (4) trivially holds under case (ii). Hence we can conclude (20) in either case directly from Theorem 4.1(i).

Conversely, under (20) if we again let $a_n \equiv 1$, we see that (7) is immediate, whence (21) follows from Theorem 4.1(ii).☐

Corollary 4.2 (Classical WLLN, Chow and Teicher, 1978, p. 125). Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables. Then

$$\frac{\sum_{k=1}^{n} X_k - nEXI(|X| \leq n)}{n} \xrightarrow{P} 0$$

iff

$$nP\{|X| > n\} = o(1).$$

Proof. By letting $b_n = n$, $n \geq 1$, the corollary follows immediately from Corollary 4.1 by either case (i) or (ii). □

Finally, we end this section with a corollary that removes the indicator functions from (6).

Corollary 4.3. Let $\{X, X_n, n \geq 1\}$ be i.i.d. $\mathcal{L}_1$ random variables. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be constants such that $a_n \neq 0$, $b_n > 0$, $n \geq 1$, $\sum_{k=1}^{n} a_k^2 = o(b_n^2)$, and $nP\{|X| > c_n\} = o(1)$. Suppose that either

(i) $c_n \uparrow$, $c_n/n \downarrow$, and $\sum_{k=1}^{n} (c_k/k)^2 = O\left(\frac{b_n^2}{\sum_{k=1}^{n} a_k^2}\right)$

or

(ii) $c_n/n \uparrow$ and $\sum_{k=1}^{n} a_k^2 = O(na_n^2)$.

If $\lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_k}{b_n} = M$ exists and is finite, then

$$\frac{\sum_{k=1}^{n} a_k X_k}{b_n} \xrightarrow{P} MEX.$$

Proof. Note that all the hypotheses of Theorem 4.1(i) are satisfied. Observe that

$$\left|\frac{\sum_{k=1}^{n} a_k EXI(|X| \le c_n)}{b_n} - MEX\right|$$

$$\le \left|\frac{\sum_{k=1}^{n} a_k EXI(|X| \le c_n)}{b_n} - MEXI(|X| \le c_n)\right| + |MEXI(|X| \le c_n) - MEX|$$

$$= |EXI(|X| \le c_n)|\left|\frac{\sum_{k=1}^{n} a_k}{b_n} - M\right| + |M||EXI(|X| \le c_n) - EX|$$

$$\le E|X|\left|\frac{\sum_{k=1}^{n} a_k}{b_n} - M\right| + |M|E|X|I(|X| > c_n).$$

Now the first term is $o(1)$ by definition of M and the fact that $X \in \mathcal{L}_1$. If $c_n \uparrow \infty$, the second term is also $o(1)$ by the Lebesgue dominated convergence theorem since $X \in \mathcal{L}_1$. Suppose $c_n \uparrow c_o < \infty$. If $c_n = c_o$ for all large n, then the hypothesis $nP\{|X| > c_n\} = o(1)$ ensures that $P\{|X| > c_o\} = 0$ and hence for all large n, $E|X|I(|X| > c_n) = E|X|I(|X| > c_o) = 0$. If $c_n < c_o$ for all large n, then the hypothesis $nP\{|X| > c_n\} = o(1)$ ensures that

$$P\{|X| > c_n\} \to P\{|X| \geq c_o\} = 0$$

implying

$$|X|I(|X| > c_n) \downarrow |X|I(|X| \geq c_o) = 0 \quad \text{a.s.}$$

Then since $X \varepsilon \mathcal{L}_1$, by the Lebesgue dominated convergence theorem $E|X|I(|X| > c_n) = o(1)$. Then applying Theorem 4.1,

$$\frac{\sum_{k=1}^{n} a_k X_k}{b_n} - MEX$$

$$= \frac{\sum_{k=1}^{n} a_k (X_k - EXI(|X| \leq c_n))}{b_n} + \frac{\sum_{k=1}^{n} a_k EXI(|X| \leq c_n)}{b_n} - MEX \xrightarrow{P} 0. \square$$

4.3 Some Interesting Examples

We conclude this chapter with three examples. The first will establish a GWLLN for a weighted sum of i.i.d. random variables whose common density function is defined as in Example 2.2. The second will be an example that establishes a GWLLN while showing that the GSLLN fails. The final example solves the Petersburg paradox in the "classical" or "weak" sense for a class of weighted i.i.d. random variables.

In Example 2.2 we have seen that if $\{X_n, n \geq 1\}$ are i.i.d. with common density function

$$f(x) = \begin{cases} \frac{e^2}{2}\left(\frac{2(\log |x|)-1}{|x|^3}\right) & \text{for } |x| \geq e, \\ 0 & \text{for } |x| < e, \end{cases}$$

then $\frac{\sum_{k=1}^n a_k X_k}{b_n} \xrightarrow{d} N(0,1)$ if $0 < b_n \uparrow \infty$ and $c_n^{(1)} \equiv b_n/|a_n| = \frac{e}{2}\sqrt{n}\log n$.

Then, in Example 3.4, we showed via Theorem 3.5 that $\frac{\sum_{k=1}^n a_k X_k}{b_n} \to 0$ a.s. if $0 < b_n \uparrow \infty$ and $c_n^{(2)} \equiv b_n/|a_n| = n^{1/p}L(n) \uparrow$ for some $0 < p < 2$ and positive slowly varying function L. Therefore, in trying to find an example of $\{a_n, n \geq 1\}$, $\{b_n, n \geq 1\}$ obeying (1), there are a few constraints that must be imposed on the sequence $\{c_n^{(3)} \equiv b_n/|a_n|, n \geq 1\}$.

Suppose that $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ are such that $0 < b_n \uparrow \infty$ and $c_n^{(3)} = \sqrt{n}\exp\{\sqrt{\log n}\}$, $n \geq 1$. A GWLLN will be established in the ensuing Example 4.1. It is interesting to compare $c_n^{(3)}$ with the $c_n^{(1)}$ and $c_n^{(2)}$ of Examples 2.2 and 3.4, respectively. For note that these $\{c_n^{(i)}, n \geq 1\}$, $1 \leq i \leq 3$, satisfy $c_n^{(1)} = o(c_n^{(3)})$ and $c_n^{(3)} = o(c_n^{(2)})$.

Before presenting Example 4.1, let us observe that Theorem 3.4 is not applicable if $c_n = \sqrt{n}\exp\{\sqrt{\log n}\}$, $n \geq 1$. To see this note for n sufficiently large that

$$\frac{1}{n}\max_{1\le k\le n} c_k^2 \sum_{j=n}^{\infty} \frac{1}{c_j^2} = e^{2\sqrt{\log n}} \sum_{j=n}^{\infty} \frac{1}{je^{2\sqrt{\log j}}}$$

$$\ge e^{2\sqrt{\log n}} \int_n^{\infty} \frac{1}{xe^{2\sqrt{\log x}}}dx \to \infty$$

(by Theorem 1(a) of Feller, 1971, p. 281)

and hence (5) (with $q = 2$) fails. We are unable to determine whether or not the GSLLN $\sum_{k=1}^{n} a_k X_k / b_n \to 0$ a.s. holds. The work of Rosalsky (1981) on generalized laws of the iterated logarithm might shed some light on this problem, but this would require some discussion.

In the following example the weights are geometrically growing.

Example 4.1. Let $\{X, X_n, n \ge 1\}$ be i.i.d. random variables with common density function

$$f(x) = \begin{cases} \frac{e^2}{2}\left(\frac{2(\log |x|)-1}{|x|^3}\right) & \text{if } |x| \ge e, \\ 0 & \text{if } |x| < e. \end{cases}$$

Then for every $a > 1$

$$\frac{\sum_{k=1}^{n} a^k X_k}{\sqrt{n}a^n e^{\sqrt{\log n}}} \xrightarrow{P} 0. \tag{22}$$

Proof. Firstly, note that $c_n \equiv \frac{b_n}{|a_n|} = \frac{\sqrt{n}a^n e^{\sqrt{\log n}}}{a^n} = \sqrt{n}e^{\sqrt{\log n}}$ and that $EXI(|X| \le c_n) = 0$, $n \ge 1$. We will verify that the conditions (2), (3), and (5) of Theorem 4.1(i)(α) obtain. Clearly $c_n \uparrow$ and $c_n/n \downarrow$ eventually. Since $a > 1$

$$\sum_{k=1}^{n} a_k^2 = \sum_{k=1}^{n} a^{2k} = \frac{a^{2(n+1)}-a^2}{a^2-1} = O(a^{2n})$$

and so

$$\frac{\sum\limits_{k=1}^{n} a_k^2}{b_n^2} = O\left(\frac{a^{2n}}{na^{2n}e^{2\sqrt{\log n}}}\right) = O\left(\frac{1}{ne^{2\sqrt{\log n}}}\right) = o(1).$$

This also shows that

$$ne^{2\sqrt{\log n}} = O\left(b_n^2 / \sum_{k=1}^{n} a_k^2\right). \tag{23}$$

Next, for n sufficiently large

$$\sum_{k=1}^{n} \left(\frac{c_k}{k}\right)^2 = \sum_{k=1}^{n} \frac{e^{2\sqrt{\log k}}}{k}$$

$$\le e^{2\sqrt{\log n}} \sum_{k=1}^{n} \frac{1}{k}$$

$$\le e^{2\sqrt{\log n}} \int_{1}^{n+1} \frac{1}{x} dx$$

$$= e^{2\sqrt{\log n}}\log(n+1)$$

$$= O(ne^{2\sqrt{\log n}}) = O\left(\frac{b_n^2}{\sum_{k=1}^{n} a_k^2}\right) \qquad \text{(recalling (23))}.$$

Finally, we need to show that (5) obtains. Recall from Example 2.2 that $P\{|X| > x\} = (\frac{e}{x})^2\log x$, $x \geq e$. Hence

$$nP\{|X| > c_n\} = \frac{ne^2}{c_n^2}\log c_n$$

$$= \frac{ne^2}{ne^{2\sqrt{\log n}}}\log(\sqrt{n}e^{\sqrt{\log n}})$$

$$= \frac{e^2}{e^{2\sqrt{\log n}}}(\frac{1}{2}\log n + \sqrt{\log n}) = o(1).$$

The conclusion (22) now follows directly from Theorem 4.1. □

The next example, which generalizes a classical example, will show that the GSLLN $\sum_{k=1}^{n} a_kX_k/b_n \to 0$ a.s. can fail while establishing the GWLLN (1). Both of these assertions will be substantiated in the example.

Example 4.2. Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables with common density function

$$f(x) = \begin{cases} \dfrac{C_\delta}{x^2(\log |x|)^\delta} & \text{if } |x| \geq e, \\ 0 & \text{if } |x| < e, \end{cases}$$

where $0 < \delta \leq 1$ and C_δ is an appropriate constant. Then for any sequence $\{a_n, n \geq 1\}$ with $0 < |a_n| \uparrow$, we have

$$\frac{\sum_{k=1}^{n} a_k X_k}{n|a_n|} \xrightarrow{P} 0 \tag{24}$$

but

$$P\{\lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_k X_k}{n|a_n|} = 0\} = 0. \tag{25}$$

Proof. We will prove (24) by applying Theorem 4.1(i)(β) with $b_n \equiv n|a_n|$, $n \geq 1$. Note that $c_n = b_n/|a_n| = n$, $n \geq 1$, and $EXI(|X| \leq n) = 0$, $n \geq 1$. Hence we need to verify (2), (4), and (5).

Clearly $c_n/n \uparrow$ and since $0 < |a_n| \uparrow$,

$$\sum_{k=1}^{n} a_k^2 \leq na_n^2 = o(b_n^2)$$

and so (2) and (4) hold. To show that (5) also holds, note that for n sufficiently large

$$nP\{|X| > c_n\} = nP\{|X| > n\}$$

$$= 2nC_\delta \int_n^\infty \frac{1}{x^2(\log x)^\delta}dx$$

$$= (1+o(1))\frac{2C_\delta}{(\log n)^\delta} \qquad \text{(by Theorem 1(a) of Feller, 1971, p. 281)} \qquad (26)$$

$$= o(1) \qquad \text{(since } \delta > 0\text{)}.$$

The conclusion (24) follows from Theorem 4.1.

We will now show that $\limsup_{n\to\infty} \dfrac{\left|\sum_{k=1}^{n} a_k X_k\right|}{n|a_n|} = \infty$ a.s. which of course implies (25). Clearly $c_n/n \uparrow$ and $b_n = O(b_{n+1})$ since $b_n = n|a_n| \uparrow$. Thus, for n_o sufficiently large

$$\sum_{n=1}^{\infty} P\{|X| > c_n\} = \sum_{n=1}^{\infty} P\{|X| > n\}$$

$$\geq \sum_{n=n_o}^{\infty} \int_{|x|>n} \frac{C_\delta}{x^2(\log |x|)^\delta}dx$$

$$= 2C_\delta \sum_{n=n_o}^{\infty} \int_n^{\infty} \frac{1}{x^2(\log x)^\delta} dx$$

$$\geq C_\delta \sum_{n=n_o}^{\infty} \frac{1}{n(\log n)^\delta} \qquad \text{(by (26))}$$

$$= \infty \qquad \text{(since } \delta \leq 1\text{)}$$

whence, via Lemma 3.8, $\limsup_{n\to\infty} \frac{|\sum_{k=1}^{n} a_k X_k|}{n|a_n|} = \infty$ a.s. and (25) follows. □

Remark. By letting $\delta = 1$ and $a_n \equiv 1$ we obtain the classical example of the WLLN holding but the SLLN failing.

We conclude with a generalization of the Petersburg game. As discussed previously, the Petersburg game involved trying to find a sequence $\{m_n, n \geq 1\}$ such that $\sum_{k=1}^{n} a_k X_k / m_n \to 1$ in some sense. The paradox occurs when the random variables are not integrable. We have already shown in Corollary 3.11 that there is not a solution to this game in the "strong" sense provided that the $\{a_n, n \geq 1\}$ satisfy certain mild conditions. We will show via an example of a particular distribution that there indeed can be a solution to the game in the "classical" or "weak" sense.

The original Petersburg game concerned tossing a fair coin. In this game a player would receive 2^j dollars if heads appears for the first time on the j^{th} toss. Thus the random variables of interest are geometric, i.e., $P\{X = 2^j\} = 2^{-j}$, $j \geq 1$. Note that even though $EX = \infty$, most people would not pay any large fixed amount of money to play since the probability of winning a large amount of money is small.

Example 4.3. Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables with $P\{X = 2^j\} = 2^{-j}$, $j \geq 1$. Then

$$\frac{\sum_{k=1}^{n} k^{\alpha} X_k}{n^{\alpha+1} \operatorname{Log} n} \xrightarrow{P} \frac{1}{\alpha+1} \qquad \text{for } \alpha > -\frac{1}{2} \tag{27}$$

(where Log n denotes the logarithm to the base 2) but

$$P\{\lim_{n\to\infty} \frac{\sum_{k=1}^{n} k^{\alpha} X_k}{m_n} = 1\} = 0 \tag{28}$$

for every $\alpha > -1$ and sequence $\{m_n, n \geq 1\}$.

Proof. Firstly, we will show that

$$P\{|X| > x\} \leq 2/x, \qquad x \geq 2. \tag{29}$$

If $2^n \leq x < 2^{n+1}$ where $n \geq 1$, then

$$P\{|X| > x\} \leq P\{X > 2^n\} = 1 - P\{X \leq 2^n\}$$

$$= 1 - \sum_{j=1}^{n} (\tfrac{1}{2})^j = \frac{1}{2^n} \leq \frac{2}{x}$$

which is tantamount to (29).

Let $b_n = n^{\alpha+1}\text{Log } n$, $n \geq 1$, and $a_n = n^\alpha$, $n \geq 1$, where $\alpha > -1/2$. Clearly $b_n > 0$ and $c_n/n = \text{Log } n \uparrow$. Also note that $\sum_{k=1}^{n} a_k^2 = \sum_{k=1}^{n} k^{2\alpha} = O(n^{2\alpha+1})$ (since $\alpha > -1/2$) and since $na_n^2 = n^{2\alpha+1}$ we observe that $\sum_{k=1}^{n} a_k^2 = O(na_n^2)$. Also note that since $b_n^2 = n^{2\alpha+2}(\text{Log } n)^2$, we have $\sum_{k=1}^{n} a_k^2 = o(b_n^2)$.

So in view of Theorem 4.1(i)(β), for a GWLLN to hold, we only need to verify (5). Now, recalling (29),

$$nP\{|X| > c_n\} = nP\{|X| > n\text{Log } n\}$$

$$\leq \frac{2n}{n\text{Log } n} = o(1)$$

whence, via Theorem 4.1,

$$\frac{\sum_{k=1}^{n} k^\alpha (X_k - EXI(|X| \leq n\text{Log } n))}{n^{\alpha+1}\text{Log } n} \xrightarrow{P} 0. \qquad (30)$$

Next, observe that

$$EXI(|X| \leq c_n) = \sum_{j=1}^{a} 2^j 2^{-j} = a$$

where a is the largest integer such that $2^a \leq c_n = n\text{Log } n$, whence $n\text{Log } n < 2^{a+1}$ and (letting [b] denote the greatest integer in b)

$$a = [\text{Log}(n\text{Log } n)] \sim \text{Log } n + \text{Log Log } n \sim \text{Log } n.$$

Thus

$$EXI(|X| \leq c_n) \sim \text{Log } n. \tag{31}$$

Also observe that

$$\frac{\sum_{k=1}^{n} k^\alpha}{n^{\alpha+1}} \sim \frac{n^{\alpha+1}}{(\alpha+1)n^{\alpha+1}} = \frac{1}{\alpha+1}. \tag{32}$$

Therefore

$$\frac{\sum_{k=1}^{n} k^\alpha X_k}{n^{\alpha+1}\text{Log } n} - \frac{1}{\alpha+1}$$

$$= \frac{\sum_{k=1}^{n} k^\alpha (X_k - EXI(|X| \leq c_n))}{n^{\alpha+1}\text{Log } n} + \frac{\sum_{k=1}^{n} k^\alpha EXI(|X| \leq c_n)}{n^{\alpha+1}\text{Log } n} - \frac{1}{\alpha+1}$$

$$= \frac{\sum_{k=1}^{n} k^{\alpha}(X_k - EXI(|X| \leq c_n))}{n^{\alpha+1} \text{Log } n} + o(1) \qquad \text{(by (31) and (32))}$$

$$\xrightarrow{P} 0 \qquad \text{(by (30))}$$

thereby proving (27).

To prove (28), we invoke Corollary 3.11 with $a_n = n^{\alpha}$, $n \geq 1$, where $\alpha > -1$. We have already observed that $E|X| = \infty$ and hence we need only verify that $\sum_{k=1}^{n} |a_k| = O(n|a_n|)$ and $n|a_n| \uparrow$. Clearly $n|a_n| = n^{\alpha+1} \uparrow$ since $\alpha > -1$. Also since $\alpha > -1$

$$\sum_{k=1}^{n} |a_k| = \sum_{k=1}^{n} k^{\alpha} = O(n^{\alpha+1}) = O(n|a_n|)$$

thereby proving (28). □

Remark. If we let $\alpha = 0$ in Example 4.3, we obtain the classical Petersburg game. Note what we have demonstrated here. We have presented a game in which there is not any normalization $\{m_n, n \geq 1\}$ for which $\sum_{k=1}^{n} k^{\alpha} X_k / m_n \to 1$ a.s. (that is, the convergence to 1 is in the "strong" sense). On the other hand, we have shown that the sequence $b_n = \frac{n^{\alpha+1}\text{Log } n}{\alpha+1}$ does indeed normalize (to 1) the sum $\sum_{k=1}^{n} k^{\alpha}X_k$ in the "classical" or "weak" sense, that is

$$\frac{(\alpha+1)\sum_{k=1}^{n} k^{\alpha}X_k}{n^{\alpha+1}\text{Log } n} \xrightarrow{P} 1.$$

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BIOGRAPHICAL SKETCH

André Bruce Adler was born on September 8, 1958, in New Rochelle, New York. His first eighteen years were spent in Baldwin and Harrison, New York. In the fall of 1976 he started his college career at the State University of New York at Binghamton. Upon completion of his Bachelor of Science (1980) degree in mathematics he moved to West Lafayette, Indiana, in pursuit of a master's degree in statistics at Purdue University. After receiving his Master of Science (1983) degree, he enrolled at the University of Florida to acquire a Ph.D. in statistics.

While at Florida, André was employed as a teaching assistant in the Department of Statistics. During those four years he taught many different courses at various mathematical levels.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

andrew Rosalsky

Andrew Rosalsky, Chairman
Associate Professor of Statistics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Ronald H. Randles

Ronald Randles
Professor of Statistics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Malay Ghosh

Malay Ghosh
Professor of Statistics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Murali Rao

Murali Rao
Professor of Mathematics

This dissertation was submitted to the Graduate Faculty of the Department of Statistics in the College of Liberal Arts and Sciences and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

May, 1987

Dean, Graduate School

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